Solutions 2

1) a) $(0,0,0,0)$

b) $f(x) = 0$

c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

d) $0x^3 + 0x^2 + 0x + 0 = 0$

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2) a) Vector space,

b) Not a vector space; Not closed under addition (Axiom 1)

$$f = x^3 + x \quad , \quad f \in P_3^*$$
$$g = -x^3 + x^2 \quad , \quad g \in P_3^*$$
$$h = f + g = x^2 + x \quad , \quad h \notin P_3^*$$

c) Not a vector space; Not closed under addition (Axiom 1)

$$f = x$$
$$g = -x$$
$$h = f + g = 0 \, , \quad h \notin \{ \phi(x) = ax + b \, ; \, a \neq 0 \}$$

d) Not a vector space; Not closed under scalar multiplication (Axiom 6)

$$\underline{u} = (1,1) \, , \quad c = -1$$
$$\underline{v} = c\,\underline{u} = (-1,1) \, , \quad v \notin \{ (x,y) \, ; \, x \geq 0 , \, y \in \mathbb{R} \}$$

e) Not a vector space; Not closed under scalar multiplication (Axiom 6)

$\underline{u} = (1,1)$, $c = -1$

$\underline{v} = c\underline{u} = (-1,-1)$, $v \notin \{(x,y): x,y \geq 0\}$

f) Vector space,

g) Vector space,

h) Not a vector space; Not closed under addition (Axiom 1)

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}, \quad \underline{\underline{B}} = \begin{pmatrix} d & e \\ f & 1 \end{pmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} = \begin{pmatrix} a+d & b+e \\ c+f & 2 \end{pmatrix}, \quad \underline{\underline{C}} \notin \left\{ \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}; a,b,c \in \mathbb{R} \right\}$$

i) Not a vector space; Not closed under addition (Axiom 1)

Singular matrix has 0 determinant,

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \underline{\underline{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ Both singular

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ Not singular

j) Vector space,

3) Let, $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\underline{\underline{B}} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $\underline{\underline{C}} = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$.

$a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{R}$

A1) $\underline{\underline{A}} + \underline{\underline{B}} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, $w, x, y, z \in \mathbb{R}$.

A2) $\underline{\underline{A}} + \underline{\underline{B}} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$, $\underline{\underline{B}} + \underline{\underline{A}} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix}$

Since,
$$a+e = e+a$$
$$b+f = f+b$$
$$c+g = g+c$$
$$d+h = h+d$$

we have, $\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}}$,

A3) $(\underline{\underline{A}} + \underline{\underline{B}}) + \underline{\underline{C}} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{pmatrix}$

$\underline{\underline{A}} + (\underline{\underline{B}} + \underline{\underline{C}}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix} = \begin{pmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{pmatrix}$

$\therefore$ Associative,

A4) $\underline{\underline{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\underline{\underline{A}} + \underline{\underline{0}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{\underline{A}}$

$\underline{\underline{0}} \in M_{22}$.

A5) $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\underline{\underline{Q}} = -\underline{\underline{A}} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$$\underline{\underline{A}} + \underline{\underline{Q}} = \begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underline{0}.$$

A6) Let $\alpha \in \mathbb{R}$,

$$\alpha\underline{\underline{A}} = \alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \quad w,x,y,z \in \mathbb{R},$$

A7) $\alpha(\underline{\underline{A}} + \underline{\underline{B}}) = \alpha\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} \alpha[a+e] & \alpha[b+f] \\ \alpha[c+g] & \alpha[d+h] \end{pmatrix}$

$$\alpha\underline{\underline{A}} + \alpha\underline{\underline{B}} = \alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \alpha\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \alpha e & \alpha f \\ \alpha g & \alpha h \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a+\alpha e & \alpha b+\alpha f \\ \alpha c+\alpha g & \alpha d+\alpha h \end{pmatrix}$$

$$= \begin{pmatrix} \alpha[a+e] & \alpha[b+f] \\ \alpha[c+g] & \alpha[d+h] \end{pmatrix}$$

$\therefore\ \alpha(\underline{\underline{A}} + \underline{\underline{B}}) = \alpha\underline{\underline{A}} + \alpha\underline{\underline{B}}$.

A8) Let, $\alpha, \beta \in \mathbb{R}$,

$$(\alpha+\beta)\underline{\underline{A}} = \begin{pmatrix} (\alpha+\beta)a & (\alpha+\beta)b \\ (\alpha+\beta)c & (\alpha+\beta)d \end{pmatrix} = \begin{pmatrix} \alpha a+\beta a & \alpha b+\beta b \\ \alpha c+\beta c & \alpha d+\beta d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \alpha \underline{\underline{A}} + \beta \underline{\underline{A}},$$

A9) $$\alpha(\beta\underline{\underline{A}}) = \alpha\left(\beta\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \alpha\begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix} = \begin{pmatrix} \alpha\beta a & \alpha\beta b \\ \alpha\beta c & \alpha\beta d \end{pmatrix}$$

$$= \begin{pmatrix} [\alpha\beta]a & [\alpha\beta]b \\ [\alpha\beta]c & [\alpha\beta]d \end{pmatrix} = (\alpha\beta)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (\alpha\beta)\underline{\underline{A}},$$

A10) $$1\underline{\underline{A}} = 1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1a & 1b \\ 1c & 1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{\underline{A}},$$

4) a) $V = \mathbb{R}^4$ is a vector space,

$$W = \{ (x_1, x_2, x_3, 0)'; \ x_1, x_2, x_3 \in \mathbb{R} \} \subseteq V$$

Let, $x_1 = x_2 = x_3 = 0 \Longrightarrow (0,0,0,0) \in W$, ✓ (contains $\underline{0}$)

Let, $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$,

$$(x_1, x_2, x_3, 0) + (y_1, y_2, y_3, 0) = (x_1+y_1, x_2+y_2, x_3+y_3, 0) \in W$$ ✓

(closed under addition)

Let, $c \in \mathbb{R}$,

$$c(x_1, x_2, x_3, 0) = (cx_1, cx_2, cx_3, 0) \in W$$ ✓

(closed under scalar multiplication)

$\therefore$ W is a subspace,

b) Let, $a = b = c = 0 \Longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ ✓

Let, $a, b, c, d, e, f \in \mathbb{R}$,

$$\begin{pmatrix} a & b \\ a+b & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & e \\ d+e & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ a+b+d+e & 0 \\ 0 & c+f \end{pmatrix} = \begin{pmatrix} w & x \\ y & 0 \\ 0 & z \end{pmatrix} \in W$$ ✓

$w, x, y, z \in \mathbb{R}$

Let $\alpha \in \mathbb{R}$, $\alpha \begin{pmatrix} a & b \\ a+b & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha(a+b) & 0 \\ 0 & \alpha c \end{pmatrix} \in W$, since $\alpha a, \alpha b, \alpha c, \alpha(a+b) \in \mathbb{R}$,

$\therefore$ W is a subspace,

5) a) $W = \{(x, y, -1) : x, y \in \mathbb{R}\}$

$(x_1, y_1, -1) + (x_2, y_2, -1) = (x_1 + x_2, y_1 + y_1, -2) \notin W.$

b) $W = \{(x, y) : x, y \in \mathbb{Q}\}$

Let $c = \sqrt{2}$.

$c(x, y) \notin \mathbb{Q}.$

c) $W = \{\underline{\underline{A}}_{mn} : |\underline{\underline{A}}_{mn}| = 0\}$

$$\left|\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\in W} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\in W}\right| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \notin W.$$

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6) a) Let $a, b = 0 \implies (0, 0, 0+0) = (0, 0, 0) = \underline{0} \in W$

$(a_1, b_1, a_1 + 2b_1) + (a_2, b_2, a_2 + 2b_2)$

$= (a_1 + a_2, b_1 + b_2, (a_1 + 2b_1) + (a_2 + 2b_2))$

$= (a_1 + a_2, b_1 + b_2, (a_1 + a_2) + 2(b_1 + b_2)) \in W$

Let $c \in \mathbb{R}$,

$c(a, b, a + 2b) = (ca, cb, c(a + 2b))$

$= (ca, cb, ca + 2cb) \in W$

$\therefore$ W is a subspace.

6) b) Let $x_1 = x_2 = 1$

$$(x_1, x_2, x_1 x_2) = (1,1,1)$$

Let $c = 2$,

$$2(x_1, x_2, x_1 x_2) = 2(1,1,1) = (2,2,2) \notin W$$

$\therefore$ not a subspace.

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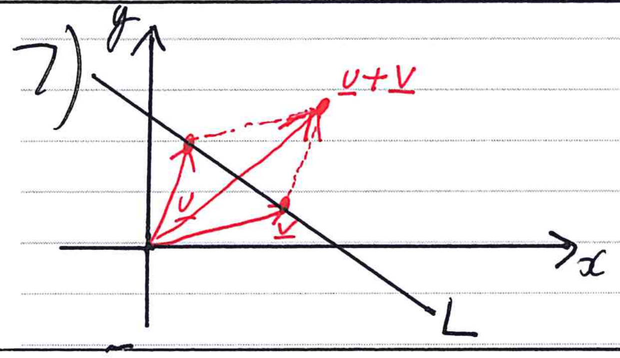
7)



$\underline{u}$ and $\underline{v}$ are on the line, L,

$(\underline{u}+\underline{v})$ is not.

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8) $$\begin{pmatrix} 2b+3c \\ -b \\ 2c \end{pmatrix} = b\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

These vectors can be expressed as linear combinations of these

spanning vectors

$$\therefore W = \text{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}\right\} \text{ is a subspace of } \mathbb{R}^3.$$

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9) a) $$W = \left\{\begin{pmatrix} 1 \\ 3a-5b \\ 3b+2a \end{pmatrix} ; a, b \in \mathbb{R}\right\}$$

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin W \quad \therefore \text{ not a subspace of } \mathbb{R}^3.$$

b) $\begin{pmatrix} 4a+3b \\ 0 \\ a+3b+c \\ 3b-2c \end{pmatrix} = a\begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 0 \\ 3 \\ 3 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}$

$\therefore W = \text{Span}\left\{ \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\}$

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10) a) $\underline{u} = (1,1,-1) \in \text{Span}\{(2,-1,3),(5,0,4)\}$?

$\Longrightarrow \quad a\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + b\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & 5 & 1 \\ -1 & 0 & 1 \\ 3 & 4 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 1 \\ 0 & 5/2 & 3/2 \\ 0 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 1 \\ 0 & 1 & 3/5 \\ 0 & 0 & -\frac{2}{5} \end{array}\right)$$

Inconsistent $\Longrightarrow$ No solution $\Longrightarrow$ $\boxed{\underline{u} \notin S.}$

$\underline{v} = \left(8, -\frac{1}{4}, \frac{27}{4}\right)$

$$\left(\begin{array}{cc|c} 2 & 5 & 8 \\ -1 & 0 & -\frac{1}{4} \\ 3 & 4 & \frac{27}{4} \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 8 \\ 0 & 5/2 & 15/4 \\ 0 & 4 & 6 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 8 \\ 0 & 1 & 3/2 \\ 0 & 1 & 3/2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 5 & 8 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array}\right)$$

$\Longrightarrow b = \frac{3}{2}, \quad a = \frac{1}{2}\left(8 - 5 \times \frac{3}{2}\right) = \frac{1}{4}$

$\left(8, -\frac{1}{4}, \frac{27}{4}\right) = \frac{1}{4}(2,-1,3) + \frac{3}{2}(5,0,4) \Longrightarrow \boxed{\underline{v} \in S.}$

$\underline{w} = (1, -8, 12)$

$$\left(\begin{array}{cc|c} 2 & 5 & 1 \\ -1 & 0 & -8 \\ 3 & 4 & 12 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 1 \\ 0 & 5/2 & -15/2 \\ 0 & 4 & -12 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$\Longrightarrow b = -3, \quad a = \frac{1}{2}\left(1 - (5 \times -3)\right) = 8$

$(1, -8, 12) = 8(2, -1, 3) - 3(5, 0, 4) \Longrightarrow \boxed{\underline{w} \in S,}$

b) $\underline{v} = (-1, 5, -6)$

$$a\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + b\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + c\begin{pmatrix} 2 \\ -12 \\ 13 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & -1 \\ 0 & 4 & -12 & 5 \\ 7 & 5 & 13 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1/2 \\ 0 & 1 & -3 & 5/4 \\ 0 & -2 & 6 & -5/2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1/2 \\ 0 & 1 & -3 & 5/4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1/2 - (5/4 + 3c) - c \\ 5/4 + 3c \\ c \end{pmatrix} = \begin{pmatrix} -7/4 - 4c \\ 5/4 + 3c \\ c \end{pmatrix} \quad (\infty \text{ solutions})$$

1 solution is when $c = 0 \Longrightarrow a = -\frac{7}{4}, \; b = \frac{5}{4}$

$$\therefore \begin{pmatrix} -1 \\ 5 \\ -6 \end{pmatrix} = -\frac{7}{4}\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + \frac{5}{4}\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \Longrightarrow \boxed{\underline{v} \in S,}$$

$\underline{v} = (-3, 15, 18)$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & -3 \\ 0 & 4 & -12 & 15 \\ 7 & 5 & 13 & 18 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3/2 \\ 0 & 1 & -3 & 15/4 \\ 0 & -2 & 6 & \frac{57}{2} \end{array}\right) \quad \text{(Inconsistent)}$$

$\therefore \boxed{\underline{v} \notin S}$

$\underline{w} = \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{2}\right)$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 1/3 \\ 0 & 4 & -12 & 4/3 \\ 7 & 5 & 13 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1/6 \\ 0 & 1 & -3 & 1/3 \\ 0 & -2 & 6 & -2/3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1/6 \\ 0 & 1 & -3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1/6 - (1/3 + 3c) - c \\ 1/3 + 3c \\ c \end{pmatrix} = \begin{pmatrix} -1/6 - 4c \\ 1/3 + 3c \\ c \end{pmatrix} \quad (\infty \text{ solutions})$$

1 solution is when $c = 0 \Longrightarrow a = -\frac{1}{6}, \ b = \frac{1}{3}$

$$\therefore \begin{pmatrix} 1/3 \\ 4/3 \\ 1/2 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \Longrightarrow \boxed{\underline{w} \in S}$$

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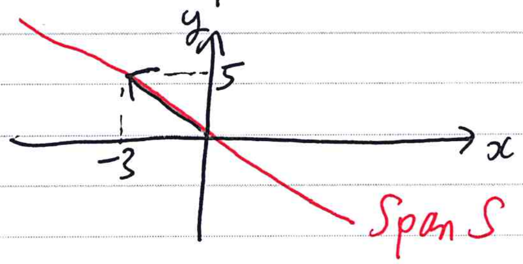
11) a) S span $\mathbb{R}^2$? $\Longrightarrow a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ v_1, v_2 \in \mathbb{R}$

$$\left(\begin{array}{cc|c} 2 & -1 & v_1 \\ 1 & 2 & v_2 \end{array}\right)$$

$\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5 \neq 0 \ \therefore$ has solution

$\Longrightarrow \boxed{S \text{ spans } \mathbb{R}^2}$

b) Vectors in span S are $a\begin{pmatrix}-3\\5\end{pmatrix}$, $a \in \mathbb{R}$,



$\boxed{\text{Span is a line,}}$

c) $\left(\begin{array}{ccc|c} 1 & -2 & 4 & v_1 \\ 3 & -6 & 12 & v_2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 4 & v_1 \\ 0 & 0 & 0 & v_2 - 3v_1 \end{array}\right)$

Has solution only when $v_1 = \dfrac{v_2}{3}$

$\therefore$ does not span $\mathbb{R}^2$, $\boxed{\text{Span is a line,}}$

d) $\left(\begin{array}{ccc|c} -1 & 4 & 1 & v_1 \\ 4 & -1 & 1 & v_2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & -1 & -v_1 \\ 0 & 15 & 5 & v_2 + 4v_1 \end{array}\right)$

$\Rightarrow$ For any $v_1, v_2 \in \mathbb{R}$ there are infinite solutions

$\therefore$ $\boxed{S \text{ spans } \mathbb{R}^2,}$

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12) a) $\left(\begin{array}{ccc|c} 4 & -1 & 2 & v_1 \\ 7 & 2 & -3 & v_2 \\ 3 & 6 & 5 & v_3 \end{array}\right)$

$$\begin{vmatrix} 4 & -1 & 2 \\ 7 & 2 & -3 \\ 3 & 6 & 5 \end{vmatrix} = 4\begin{vmatrix} 2 & -3 \\ 6 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -3 \\ 3 & 5 \end{vmatrix} + 2\begin{vmatrix} 7 & 2 \\ 3 & 6 \end{vmatrix}$$

$$= 4(28) + (44) + 2(36)$$

$$= 228 \neq 0 \Rightarrow \text{has solution}$$

$\therefore$ $\boxed{S \text{ spans } \mathbb{R}^3,}$

b) $\left(\begin{array}{cc|c} -2 & 4 & u_1 \\ 5 & 6 & u_2 \\ 0 & 3 & u_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & -u_1/2 \\ 0 & 16 & u_2 + 5u_1/2 \\ 0 & 3 & u_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & -u_1/2 \\ 0 & 1 & \frac{1}{16}(u_2 + 5u_1/2) \\ 0 & 1 & \frac{1}{3}(u_3) \end{array}\right)$

Has solution only when,

$$u_3 = \frac{3}{16}\left(u_2 + \frac{5u_1}{2}\right)$$

$\therefore$ does not span $\mathbb{R}^3$. $\boxed{\text{Span is a plane.}}$

c) $\left(\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ -2 & 0 & 2 & u_2 \\ 0 & 1 & 0 & u_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 1 & 0 & u_3 \\ 0 & 0 & 0 & u_2 + 2u_1 \end{array}\right)$

Has solution only when,

$$u_2 = -2u_1$$

$\therefore$ does not span $\mathbb{R}^3$, $\boxed{\text{Span is a plane,}}$

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13) a) $\nexists a \in \mathbb{R}: a\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ $\therefore$ Linearly independent.

b) Linearly dependent $\because$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

c) $\left(\begin{array}{ccc|c} -4 & 1 & 6 & 0 \\ -3 & -2 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{array}\right)$

$\begin{vmatrix} -4 & 1 & 6 \\ -3 & -2 & 0 \\ 4 & 3 & 0 \end{vmatrix} = 6\begin{vmatrix} -3 & -2 \\ 4 & 3 \end{vmatrix} = -6 \neq 0$ $\therefore$ unique solution.

For homogeneous solution it is $\underline{0}$.

Only trivial solution $\therefore$ linearly independent.

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d) $\left(\begin{array}{ccc|c} 4 & 1 & 3 & 0 \\ -3 & 8 & -2 & 0 \\ 6 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 1 & 3 & 0 \\ 0 & 35/4 & 1/4 & 0 \\ 0 & 3/2 & -11/2 & 0 \\ 0 & 1/2 & -3/2 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 4 & 1 & 3 & 0 \\ 0 & 1 & 1/35 & 0 \\ 0 & 0 & -194/35 & 0 \\ 0 & 0 & -53/35 & 0 \end{array}\right) \Longrightarrow$ Only has trivial solution

$\therefore$ linearly independent.

e) $\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right) \Longrightarrow$ Only has trivial solution

$\therefore$ linearly independent.

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14) a) $\left(\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 4 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & 7/3 & -8/3 & 0 \end{array}\right) \Longrightarrow \infty$ solutions

$\therefore$ linearly dependent.

$$7b = 8c$$

$$3a = b - 2c = \frac{8}{7}c - 2c = -\frac{6}{7}c$$

$$\therefore a = -\frac{2}{7}c$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c\begin{pmatrix} -2/7 \\ 8/7 \\ 1 \end{pmatrix}$$

Taking $c = 7 \Rightarrow a = -2,\ b = 8$

$$-2\begin{pmatrix} 3 \\ 4 \end{pmatrix} + 8\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \underline{0} \Longrightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{7}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

b) $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \infty$ solutions

$\therefore$ linearly dependent.

$$b = c \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$a = -b = -c$$

Taking $c = 1 \Rightarrow a = -1,\ b = 1$

$$-\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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15) a) $\nexists a \in \mathbb{R} :\ a\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\therefore$ linearly independent.

2 linearly independent vectors from $\mathbb{R}^2$ span $\mathbb{R}^2$

$\therefore$ S is a basis for $\mathbb{R}^2$.

b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 2 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

Only trivial solution $\therefore$ linearly independent.

3 linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$

$\therefore$ S is a basis for $\mathbb{R}^3$.

c) $\left(\begin{array}{ccc|c} 0 & 4 & -8 & 0 \\ 3 & 0 & 15 & 0 \\ -2 & 3 & -16 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

∞ solutions $\therefore$ linearly dependent

$\Rightarrow$ S is not a basis for $\mathbb{R}^3$.

d) $\left(\begin{array}{cccc|c} -1 & 2 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

Only trivial solution $\therefore$ linearly independent.

$\Rightarrow$ S is a basis for $\mathbb{R}^4$.

e) $a\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + b\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \underline{0}$

$\Rightarrow$ $2a + b = 0$

$4b + c + d = 0$

$3c + 2d = 0$

$3a + b + 2c = 0$

$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 3 & 1 & 2 & 0 & 0 \end{array}\right) \Rightarrow \begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 3 & 1 & 2 & 0 \end{vmatrix} = -31 \neq 0$

$\therefore$ unique solution (trivial) $\Rightarrow$ linearly independent.

$\Rightarrow$ S is a basis for M_{22}.

16) a) $\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ trivial solution only,

$\Rightarrow$ Linearly independent $\therefore$ S is a basis for $\mathbb{R}^3$,

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 8 \\ 3 & 3 & 0 & 3 \\ 2 & 2 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & -3 \\ 0 & 2 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\therefore \begin{pmatrix} 8 \\ 3 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

b) $\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & -17 & 0 \\ 0 & 0 & -26 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_2, x_3 = 0, \quad x_1 \in \mathbb{R} \Rightarrow \infty$ solutions

$\Rightarrow$ linearly dependent $\therefore$ S is not a basis for $\mathbb{R}^3$,

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17) If $\underline{\underline{A}}\,\underline{w} = \underline{0}$ then $\underline{w} \in \text{Nul}\,\underline{\underline{A}}$,

$$\begin{pmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \underline{w} \in \text{Nul}\,\underline{\underline{A}},$$

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18) a) $a\begin{pmatrix} -1 \\ 4 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 4 & 0 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & 4 \end{array}\right) \Rightarrow a = 1,\ b = 2$$

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$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \therefore \ \underline{b} \in \operatorname{col} \underline{\underline{A}} .$$

b) $$a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{array}\right)$$

Inconsistent $\Rightarrow$ No solution $\therefore \ \underline{b} \notin \operatorname{col} \underline{\underline{A}}$.