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1) a) Find all pairs of linearly independent vectors.

By inspection: $\{(1,0),(0,1)\}$, $\{(0,1),(1,1)\}$

$\{(1,0),(1,1)\}$

b) Find all sets of 3 vectors which are linearly independent.

Possible sets:

$$S_1=\{(1,3,-2),(-4,1,1),(-2,7,-3)\}$$

$$S_2=\{(1,3,-2),(-4,1,1),(2,1,1)\}$$

$$S_3=\{(1,3,-2),(-2,7,-3),(2,1,1)\}$$

$$S_4=\{(-4,1,1),(-2,7,-3),(2,1,1)\}$$

Check linear independence:

S_1: $\begin{vmatrix} 1 & -4 & -2 \\ 3 & 1 & 7 \\ -2 & 1 & -3 \end{vmatrix} = 0 \therefore$ linearly dependent.

S_2: $\begin{vmatrix} 1 & -4 & 2 \\ 3 & 1 & 1 \\ -2 & 1 & 1 \end{vmatrix} = 30 \neq 0 \therefore$ linearly independent.

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S_3: $\begin{vmatrix} 1 & -2 & 2 \\ 3 & 7 & 1 \\ -2 & -3 & 1 \end{vmatrix} = 30 \neq 0$ $\therefore$ linearly independent.

S_4: $\begin{vmatrix} -4 & -2 & 2 \\ 1 & 7 & 1 \\ 1 & -3 & 1 \end{vmatrix} = -60 \neq 0$ $\therefore$ linearly independent.

$\therefore$ S_2, S_3 & S_4 are bases for $\mathbb{R}^3$.

c) Any linearly independent vector in $\mathbb{R}^2$ will do.

$\{(1,1),(1,0)\}$ or $\{(1,1),(0,1)\}$

or $\{(1,1),(2,3)\}$ etc.

d) Any linearly independent vector in $\mathbb{R}^3$.

$\{(1,0,2),(0,1,1),(1,1,1)\}$ etc.

e) We can either form rows from vectors then row reduce $\Longrightarrow$ non-zero rows form basis

or

Form columns from vectors then row reduce $\Longrightarrow$ take linearly independent columns.

Double A

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I will use the second method.

$$\begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ -2 & 2 & -8 & 10 & -6 \\ 3 & 3 & 0 & 3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 2 & -4 & 14 & 0 \\ 0 & 3 & -6 & -3 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 16 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1^{st}, 2^{nd}, 4^{th} & 5^{th} columns are linearly independent.

$\therefore \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 10 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -6 \\ 9 \end{pmatrix} \right\}$ is a basis.

f) I will use the row method.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 3 & -1 & 2 & -1 \\ 5 & -3 & 3 & -4 \\ 2 & -1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & -1 & 2 & -4 \\ 0 & -3 & 3 & -9 \\ 0 & -1 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Moving to bottom & shifting other rows up gives an echelon form

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced echelon form

We can use either form to make a basis.

Basis 1: $\{(1,0,0,1),(0,1,-2,4),(0,0,-3,3),(0,0,0,1)\}$

or

Basis 2: $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

g) $\{\sin t, \sin 2t, \sin t \cos t\} = \{\sin t, \sin 2t, \frac{1}{2}\sin 2t\}$

By inspection $\{\sin t, \sin 2t\}$ spans the same subspace & is linearly independent $\therefore$ a basis.

h) Using the column technique:

$$(\underline{v}_1^T, \underline{v}_2^T, \underline{v}_3^T) = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 4 \\ 0 & -1 & 1 \\ -1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Linearly independent columns

$\therefore \{\underline{v}_1, \underline{v}_2\}$ is a basis for H.

$$(\underline{v}_1^T, \underline{v}_2^T, \underline{v}_3^T) = \begin{pmatrix} -2 & 2 & -1 \\ -2 & 3 & 4 \\ -1 & 2 & 6 \\ 3 & -6 & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 1 & 13/2 \\ 0 & -3 & -7/2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -2 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

3 Linearly independent columns

$\therefore \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a basis for K.

$$(\underline{u}_1^T, \underline{u}_2^T, \underline{u}_3^T, \underline{v}_1^T, \underline{v}_2^T, \underline{v}_3^T) = \begin{pmatrix} 1 & 0 & 3 & -2 & 2 & -1 \\ 2 & 2 & 4 & -2 & 3 & 4 \\ 0 & -1 & 1 & -1 & 2 & 6 \\ -1 & 1 & -4 & 3 & -6 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & -2 & 2 & -1 \\ 0 & 2 & -2 & 2 & -1 & 6 \\ 0 & -1 & 1 & -1 & 2 & 6 \\ 0 & 1 & -1 & 1 & -4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & -2 & 2 & -1 \\ 0 & 2 & -2 & 2 & -1 & 6 \\ 0 & 0 & 0 & 0 & 3/2 & 9 \\ 0 & 0 & 0 & 0 & 0 & 15 \end{pmatrix}$$

Linearly independent columns

$\therefore \{\underline{u}_1, \underline{u}_2, \underline{v}_2, \underline{v}_3\}$ is a basis for $(H+K)$.

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2) a) $\begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

Linearly independent by inspection,

Basis vectors, dimension 2,

b) $\begin{pmatrix} p-2q \\ 2p+5r \\ -2q+2r \\ -3p+6r \end{pmatrix} = p\begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix} + q\begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 5 \\ 2 \\ 6 \end{pmatrix}$

Check linear independence:

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 5 \\ 0 & -2 & 2 \\ -3 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 5 \\ 0 & -2 & 2 \\ 0 & -6 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 9/2 \\ 0 & 0 & 0 \end{pmatrix}$$

3 Linearly independent vectors ($\because$ 3 pivot columns)

$\therefore \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 2 \\ 6 \end{pmatrix} \right\}$ is a basis,

Dimension 3.

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c) $a - 3b = 0$
$b - 2c = 0$ $\Longrightarrow \underline{\underline{A}} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{pmatrix}$
$2b - c = 0$

The subspace is Nul $\underline{\underline{A}}$,

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \Longrightarrow \text{only trivial solution,}$$

$\therefore (a, b, c) = (0, 0, 0)$

The subspace is $\{ \underline{0} \}$ with dimension 0,

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d) $$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3b - c \\ b \\ c \\ d \end{pmatrix} = b \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Linearly independent by inspection.

$\therefore$ Basis is $\{ (3,1,0,0), (-1,0,1,0), (0,0,0,1) \}$

Dimension 3.

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e) $$\begin{pmatrix} a \\ b \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Linearly independent by inspection,

Basis vectors, dimension 2,

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f) $(-2, 10) = -2(1, -5)$ Linearly dependent

Since $(1,-5)$ & $(-3,15)$ are linearly independent they form a basis,

Dimension 2,

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g) Column method,

$$\begin{pmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & -5 & 5 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ \underline{0} & \underline{0} & 0 & \underline{2} \end{pmatrix}$$

Linearly independent columns

$\therefore$ Basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} \right\}$

Dimension 3,

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h) Column method.

$$\begin{pmatrix} 1 & -3 & -2 & -3 \\ -2 & -8 & 3 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -2 & -3 \\ 0 & -12 & -1 & -1 \\ \underline{0} & \underline{0} & \underline{5} & 5 \end{pmatrix}$$

$\therefore$ Basis is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -8 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\}$

Dimension 3.

3) a) $\begin{pmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

x_3 free, $x_2 = 0$, $x_1 = x_3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow$ Basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Nullity 1.

b) $\begin{pmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \end{pmatrix}$

x_3, x_4 free, $x_2 = -\frac{1}{4}x_3 - x_4$, $x_1 = -\frac{1}{4}x_3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}x_3 \\ -\frac{1}{4}x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

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A basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, nullity 2,

c) $\begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow$ x_2, x_3 free

$x_1 = 3x_2 - x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

A basis is $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, nullity 2,

4) a) Already echelon form.

Rank 2, row space basis $\{(1,0), (0,2)\}$

column space basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$

b) $(1\ 2\ 3)$ is echelon form,

Rank 1, row space basis $\{(1,2,3)\}$

column space basis $\{(1)\}$

c) $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 14 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$

Rank 2, row space basis $\left\{ (1, 0, \tfrac{1}{2}), (0, 1, -\tfrac{1}{2}) \right\}$

column space basis $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}$

d) $\begin{pmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{pmatrix} \sim \begin{pmatrix} 4 & 20 & 31 \\ 0 & -35 & -105/2 \\ 0 & -21 & -63/2 \end{pmatrix} \sim \begin{pmatrix} 4 & 20 & 31 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}$

Rank 2, row space basis $\left\{ (4, 20, 31), (0, 1, \tfrac{3}{2}) \right\}$

Column space basis $\left\{ \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 20 \\ -5 \\ -11 \end{pmatrix} \right\}$

e) $\begin{pmatrix} -2 & -4 & 4 & 5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{pmatrix} \sim \begin{pmatrix} -2 & -4 & 4 & 5 \\ 0 & 0 & 0 & 7/2 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Rank 2

row space basis $\left\{ (1, 2, -2, 0), (0, 0, 0, 1) \right\}$

Column Space basis $\left\{ \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \\ 9 \end{pmatrix} \right\}$

f) $\begin{pmatrix} 2 & 4 & -2 & 1 & 1 \\ 2 & 5 & 4 & -2 & 2 \\ 4 & 3 & 1 & 1 & 2 \\ 2 & -4 & 2 & -1 & 1 \\ 0 & 1 & 4 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -2 & 1 & 1 \\ 0 & 1 & 6 & -3 & 1 \\ 0 & -5 & 5 & -1 & 0 \\ 0 & -8 & 4 & -2 & 0 \\ 0 & 1 & 4 & 2 & -1 \end{pmatrix}$

$\sim \begin{pmatrix} 2 & 4 & -2 & 1 & 1 \\ 0 & 1 & 6 & -3 & 1 \\ 0 & 0 & 35 & -16 & 5 \\ 0 & 0 & 52 & -26 & 8 \\ 0 & 0 & -2 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -2 & 1 & 1 \\ 0 & 1 & 6 & -3 & 1 \\ 0 & 0 & 35 & -18 & 5 \\ 0 & 0 & 0 & -78/35 & 4/7 \\ 0 & 0 & 0 & \frac{143}{35} & \frac{-12}{7} \end{pmatrix}$

$\sim \begin{pmatrix} 2 & 4 & -2 & 1 & 1 \\ 0 & 1 & 6 & -3 & 1 \\ 0 & 0 & 35 & -18 & 5 \\ 0 & 0 & 0 & 1 & -10/39 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Rank 5, row space basis $\{(1,0,0,0,0),(0,1,0,0,0),$
$(0,0,1,0,0),(0,0,0,1,0),$
$(0,0,0,0,1)\}$

Column space basis are the columns of the original matrix.

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5) a) $\underline{b}$ is in column space if $\underline{\underline{A}}\underline{x} = \underline{b}$ has a solution.

$$\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 4 & 0 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) \quad \therefore \underline{b} \in \text{col } \underline{\underline{A}},$$

$$\underline{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

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b) $$\left(\begin{array}{cc|c} -1 & 2 & 2 \\ 2 & -4 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & -2 \\ 0 & 0 & 8 \end{array}\right)$$ No solution

$\therefore \underline{b} \notin \text{col } \underline{\underline{A}}$.

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c) $$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & 2 \\ 2 & 0 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 5 & 0 & 3 \\ 0 & -6 & 1 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -7/5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -7/5 \end{array}\right) \quad \therefore \underline{b} \in \text{col } \underline{\underline{A}}$$

$$\underline{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = -\frac{4}{5}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \frac{7}{5}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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d) $$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

No solution $\therefore \underline{b} \notin \text{col } \underline{\underline{A}}$.

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6) a)
$$\begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank 2, row space basis $\left\{ (1,0,-1,5), (0,1,-\frac{5}{2},3) \right\}$

$\Rightarrow$ Nullity $= 4-2 = 2$,

Column space basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} \right\}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 - 5x_4 \\ \frac{5}{2}x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Basis for Nul $\underline{\underline{A}}$.

b)
$$\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & -2 & 2 & -7 \\ 0 & 0 & -9 & 9 & -9 \\ 0 & 0 & -12 & 12 & -6 \end{pmatrix}$$

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$$\sim \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 45/2 \\ 0 & 0 & 0 & 0 & 36 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank $= 3$, row space basis $\{(1,3,0,3,0), (0,0,1,-1,0), (0,0,0,0,1)\}$

$\Rightarrow$ Nullity $= 5-3 = 2$

column space basis $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -3 \\ 0 \end{pmatrix} \right\}$

$$\underline{x} = \begin{pmatrix} -3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Basis for Nul $\underline{A}$,

c) $\begin{pmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ -2 & -3 & 6 & -3 & 0 & -6 \\ 4 & 9 & -12 & 9 & 3 & 12 \\ -2 & 3 & 6 & 3 & 3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -3 & 3 & 3/2 & 3 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & -3 & 0 & -3 & -3 & 0 \\ 0 & 9 & 0 & 9 & 6 & 0 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 3 & -3 & 3 & 3/2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Rank $= 3$, row space basis $\{(1, 0, -3, 0, 0, 3),$
$\Rightarrow$ Nullity $= 6 - 3 = 3$

$(0, 1, 0, 1, 1, 0)$

$(0, 0, 0, 0, 1, 0)\}$

column space basis $\left\{ \begin{pmatrix} 2 \\ -2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ -3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 3 \end{pmatrix} \right\}$

$$x_5 = 0, \quad \underline{x} = \begin{pmatrix} 3x_3 - 3x_6 \\ -x_4 \\ x_3 \\ x_4 \\ 0 \\ x_6 \end{pmatrix} = x_3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis for Nul $\underline{A}$,

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7) a) $[x]_B = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow 4\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

Standard basis: $S = \{(1,0), (0,1)\}$

$\begin{pmatrix} 8 \\ -3 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow [x]_S = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

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b) $[x]_B = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \Rightarrow 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$

Standard basis: $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\Rightarrow [x]_S = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$

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c) $[x]_B = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -1 \end{pmatrix} \Rightarrow 1\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

Standard basis: $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$\Rightarrow [x]_S = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

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8) a) $[x]_S = \begin{pmatrix} 12 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow c_1 = 3, \ c_2 = 2$

$\therefore [x]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b) $\begin{pmatrix} 3 \\ 19 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ 11 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 7 \\ 0 \\ 10 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 8 & 7 & 1 & 3 \\ 11 & 0 & 4 & 19 \\ 0 & 10 & 6 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 8 & 7 & 1 & 3 \\ 0 & -77/8 & 21/8 & 119/8 \\ 0 & 10 & 6 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 8 & 7 & 1 & 3 \\ 0 & 1 & -3/11 & -17/11 \\ 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\therefore [x]_B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

c) $\begin{pmatrix} 11 \\ 18 \\ -7 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -11 \\ 0 \\ 11 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 9 \\ 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 4 & -11 & 0 & 11 \\ 3 & 0 & 9 & 18 \\ 3 & 11 & 2 & -7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & -11 & 0 & 11 \\ 0 & 33/4 & 9 & 39/4 \\ 0 & 77/4 & 2 & -61/4 \end{array}\right)$$

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$$\sim\left(\begin{array}{ccc|c} 4 & -11 & 0 & 11 \\ 0 & 1 & -\frac{12}{11} & \frac{13}{11} \\ 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\therefore [x]_B = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

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9) a) $\underline{\underline{B}} \to \underline{\underline{B}}'$; $(\underline{\underline{B}}' | \underline{\underline{B}}) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & 1 \end{array}\right)$$

Transition matrix: $\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$

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b) $\left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 3 \end{array}\right)$ Already in correct form

$\therefore$ transition matrix is $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$

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c) $\left(\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 8 & 12 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{12} \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{12} \end{array}\right)$ $\therefore$ transition matrix: $\begin{pmatrix} 1 & 2 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$

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10) a) $\underline{B' \to B}$:

$$\left(\begin{array}{cc|cc} 1 & -2 & -12 & -4 \\ 3 & -2 & 0 & 4 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & 9 & 4 \end{array}\right)$$

Transition

$\underline{B \to B'}$:

$$\left(\begin{array}{cc|cc} -12 & -4 & 1 & -2 \\ 0 & 4 & 3 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{2} \end{array}\right)$$

Transition

$$\Rightarrow \begin{pmatrix} 6 & 4 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{3}{4} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$[x]_{B'} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \therefore [x]_B = \begin{pmatrix} 6 & 4 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

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b) $\underline{B' \to B}$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{5}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{3} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{2} & \frac{5}{4} \end{array}\right)$$

Transition

$\underline{B \to B'}$:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 5 & 1 \\ 0 & 1 & 0 & -7 & -10 & -1 \\ 0 & 0 & 1 & -2 & -2 & 0 \end{array}\right)$$

Transition

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{5}{4} \\ -\frac{1}{2} & -\frac{1}{3} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 4 & 5 & 1 \\ -7 & -10 & -1 \\ -2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$[x]_{B'} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \therefore [x]_B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{5}{4} \\ -\frac{1}{2} & -\frac{1}{3} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{11}{4} \\ -\frac{9}{4} \\ \frac{5}{4} \end{pmatrix}$$