Practice Problems 3

1) a) Find all subsets of the following set that form a basis for $\mathbb{R}^2$.

$$S = \{(1,0),(0,1),(1,1)\}$$

b) Find all subsets of the following set that form a basis for $\mathbb{R}^3$.

$$S = \{(1,3,-2),(-4,1,1),(-2,7,-3),(2,1,1)\}$$

c) Find a basis for $\mathbb{R}^2$ that includes the vector $(1,1)$.

d) Find a basis for $\mathbb{R}^3$ that includes the set $\{(1,0,2),(0,1,1)\}$.

e) Find a basis for the space spanned by the following set.

$$S = \{(1,0,-2,3),(0,1,2,3),(2,-2,-8,0),(2,-1,10,3),(3,-1,-6,9)\}$$

f) Find a basis for the space spanned by the following set.

$$S = \{(1,0,0,1),(-2,0,0,2),(3,-1,2,-1),(5,-3,3,-4),(2,-1,2,0)\}$$

g) Find a basis for the subspace of real-valued functions spanned by the following set.

$$\{\sin t\,, \sin 2t\,, \sin t \cos t\}$$

h) Let $H = Span\{\mathbf{u_1},\mathbf{u_2},\mathbf{u_3}\}$ and $K = Span\{\mathbf{v_1},\mathbf{v_2},\mathbf{v_3}\}$, where,

$$\mathbf{u_1} = (1,2,0,-1), \mathbf{u_2} = (0,2,-1,1), \mathbf{u_3} = (3,4,1,-4)$$
$$\mathbf{v_1} = (-2,-2,-1,3), \mathbf{v_2} = (2,3,2,-6), \mathbf{v_3} = (-1,4,6,-2)$$

Find bases for H, K and $H + K$.

2) For each subspace find a basis and state the dimension.

a) $\{(s-2t, s+t, 3t): s,t \in \mathbb{R}\}$
b) $\{(p-2q, 2p+5r, -2q+2r, -3p+6r): p,q,r \in \mathbb{R}\}$
c) $\{(a,b,c): a-3b=0, b-2c=0, 2b-c=0\}$
d) $\{(a,b,c,d): a-3b+c=0\}$
e) All vectors in $\mathbb{R}^3$ whose first and third entries are equal.
f) $Span\{(1,-5),(-2,10),(-3,16)\}$
g) $Span\{(1,0,2),(3,1,1),(-2,-1,1),(5,2,2)\}$

h) $Span\{(1,-2,0),(-3,-6,0),(-2,3,5),(-3,5,5)\}$

3) Find a basis for the solution space of the following homogeneous systems (the null space) then state the dimension of the space (nullity).

a)
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

b)
$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

c)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

4) Find the rank of the following matrices and bases for both the row and column spaces.

a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 2 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{pmatrix}$

e) $\begin{pmatrix} -2 & -4 & 4 & 5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{pmatrix}$

f) $\begin{pmatrix} 2 & 4 & -2 & 1 & 1 \\ 2 & 5 & 4 & -2 & 2 \\ 4 & 3 & 1 & 1 & 2 \\ 2 & -4 & 2 & -1 & 1 \\ 0 & 1 & 4 & 2 & -1 \end{pmatrix}$

5) Determine whether **b** is in the column space of **A**. If it is then write it as a linear combination of the column vectors in **A.**

a) $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 4 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

c) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

d) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

6) Find the rank and nullity of the following matrices then find bases for the column, row and null spaces.

a) $\begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ -2 & -3 & 6 & -3 & 0 & -6 \\ 4 & 9 & -12 & 9 & 3 & 12 \\ -2 & 3 & 6 & 3 & 3 & -6 \end{pmatrix}$

7) Given a non-standard basis and coordinates relative to that basis, find the coordinate vector relative to the standard basis in $\mathbb{R}^n$.

a) $B = \{(2,-1),(0,1)\}$, $[\mathbf{x}]_B = (4,1)^T$

b) $B = \{(1,0,1),(1,1,0),(0,1,1)\}$, $[\mathbf{x}]_B = (2,3,1)^T$

c) $B = \{(0,0,0,1),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$, $[\mathbf{x}]_B = (1,-2,3,-1)^T$

8) Find the coordinates of $\mathbf{x}$ in $\mathbb{R}^n$ in the standard basis relative to the specified basis, B.

a) $B = \{(4,0),(0,3)\}$, $\mathbf{x} = (12,6)^T$

b) $B = \{(8,11,0),(7,0,10),(1,4,6)\}$, $\mathbf{x} = (3,19,2)^T$

c) $B = \{(4,3,3),(-11,0,11),(0,9,2)\}$, $\mathbf{x} = (11,18,-7)^T$

9) Find the transition matrix from B to B'.

a) $B = \{(1,0),(0,1,)\}$, $B' = \{(2,4),(1,3)\}$

b) $B = \{(2,4),(-1,3)\}$, $B' = \{(1,0),(0,1)\}$

c) $B = \{(1,0,0),(0,1,0),(0,0,1)\}$, $B' = \{(1,0,0),(0,2,8),(6,0,12)\}$

10) Find the transition matrix from B' to B, the transition matrix from B to B', verify that they are inverses of each other, then find $[\mathbf{x}]_B$ when provided with $[\mathbf{x}]_{B'}$.

a) $B = \{(1,3),(-2,-2)\}$, $B' = \{(-12,0),(-4,4)\}$, $[\mathbf{x}]_{B'} = (-1,3)^T$

b) $B = \{(1,0,2),(0,1,3),(1,1,1)\}$, $B' = \{(2,1,1),(1,0,0),(0,2,1)\}$, $[\mathbf{x}]_{B'} = (1,2,-1)^T$