SUBJECT: Solutions 4 NO: DATE: / /

1) a) $\|\underline{v}\| = \sqrt{16+9} = \sqrt{25} = 5$

b) $\|\underline{v}\| = \sqrt{1+4+4} = \sqrt{9} = 3$

c) $\|\underline{v}\| = \sqrt{4+0+25+25} = \sqrt{54} = 3\sqrt{6}$

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2) a) $\|\underline{u}\| = \sqrt{0+16+9} = \sqrt{25} = 5$

$\|\underline{v}\| = \sqrt{1+4+1} = \sqrt{6}$

$\|\underline{u} + \underline{v}\| = \sqrt{(0+1)^2 + (4-2)^2 + (3+1)^2} = \sqrt{1+4+16} = \sqrt{21}$

b) $\|\underline{u}\| = \sqrt{0+1+1+4} = \sqrt{6}$

$\|\underline{v}\| = \sqrt{1+1+9+0} = \sqrt{11}$

$\|\underline{u} + \underline{v}\| = \sqrt{(0+1)^2 + (1+1)^2 + (-1+3)^2 + (2+0)^2} = \sqrt{1+4+4+4} = \sqrt{13}$

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3) a) $\underline{u} = (-5, 12)$, $\|\underline{u}\| = \sqrt{25+144} = \sqrt{169} = 13$

Unit vector: $\hat{\underline{u}} = \dfrac{\underline{u}}{\|\underline{u}\|} = \dfrac{1}{13}(-5, 12)$

b) $\underline{u} = (1, 0, 2, 2)$, $\|\underline{u}\| = \sqrt{1+0+4+4} = \sqrt{9} = 3$

Unit vector: $\hat{\underline{u}} = \dfrac{\underline{u}}{\|\underline{u}\|} = \dfrac{1}{3}(1, 0, 2, 2)$

SUBJECT: NO: DATE: / /

4) $\|c(1,2,3)\| = 7$

$|c|\,\|(1,2,3)\| = 7$

$|c|\sqrt{1+4+9} = 7$

$|c| = \dfrac{7}{\sqrt{14}} \quad \therefore \quad c = \pm\dfrac{7}{\sqrt{14}}$

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5) a) Unit vector in direction of $\underline{u}$ has length 1,

$$\hat{\underline{u}} = \frac{1}{\sqrt{3+9+0}}(\sqrt{3},3,0) = \frac{1}{\sqrt{12}}(\sqrt{3},3,0) = \frac{1}{2\sqrt{3}}(\sqrt{3},3,0)$$

Vector in direction of $\underline{u}$ with length 2 is then:

$$\underline{w} = 2 \times \frac{1}{2\sqrt{3}}(\sqrt{3},3,0) = (1,\sqrt{3},0)$$

b) $$\hat{\underline{u}} = \frac{1}{\sqrt{0+4+1+4}}(0,2,1,-2) = \frac{1}{3}(0,2,1,-2)$$

Vector in direction of $\underline{u}$ with length 3 is then:

$$\underline{w} = 3 \times \frac{1}{3}(0,2,1,-2) = (0,2,1,-2)$$

SUBJECT: NO: DATE: / /

6) a) $d(\underline{u},\underline{v}) = \|\underline{u}-\underline{v}\|$

$= \|(1,1,2)-(-1,3,0)\|$

$= \|(2,-2,2)\| = \sqrt{12} = 2\sqrt{3}$

b) $d(\underline{u},\underline{v}) = \|(0,1,2,3)-(1,0,4,-1)\|$

$= \|(-1,1,-2,4)\| = \sqrt{22}$

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7) $\underline{u}\cdot\underline{v} = (-1\times 1)+(1\times -3)+(2\times -2) = -8$

$\underline{u}\cdot\underline{u} = (-1)^2+(1)^2+(2)^2 = 6$

$\|\underline{u}\| = \underline{u}\cdot\underline{u} = 6$

$(\underline{u}\cdot\underline{v})\underline{v} = -8(1,-3,-2) = (-8,24,16)$

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8) $(\underline{u}+\underline{v})\cdot(2\underline{u}-\underline{v}) = \underline{u}\cdot(2\underline{u}-\underline{v}) + \underline{v}\cdot(2\underline{u}-\underline{v})$

$= 2\underline{u}\cdot\underline{u} - \underline{u}\cdot\underline{v} + 2\underline{u}\cdot\underline{v} - \underline{v}\cdot\underline{v}$

$= 2\underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} - \underline{v}\cdot\underline{v} = 2\times 4 - 5 - 10$

$= -7$

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9) a) $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\|\underline{v}\|} = \dfrac{(1\times 2)+(1\times 1)+(1\times -1)}{\sqrt{1+1+1}\sqrt{4+1+1}} = \dfrac{2}{\sqrt{3}\sqrt{6}} = \dfrac{2}{\sqrt{18}}$

$= \dfrac{2}{3\sqrt{2}}$

$\therefore \theta = \cos^{-1}\left(\dfrac{2}{3\sqrt{2}}\right) = 1.08^c = 61.87^\circ$

SUBJECT:.................... NO:.......... DATE: ..../..../....

b) $\cos\theta = \dfrac{0+3+0+3}{\sqrt{2}\sqrt{36}} = \dfrac{1}{\sqrt{2}}$ $\therefore \theta = \dfrac{\pi}{4}^c = 45^\circ$

c) $\cos\theta = \dfrac{-1+12-5-6+0}{\sqrt{15}\sqrt{55}} = 0$ $\therefore \theta = \dfrac{\pi}{2}^c = 90^\circ$

10) a) $\underline{v} = (v_1, v_2)$ is orthogonal if $\underline{u} \cdot \underline{v} = 0$,

$$(0,5)\cdot(v_1, v_2) = 0$$

$$0v_1 + 5v_2 = 0 \implies v_2 = 0$$

$$\therefore \underline{v} = (t, 0), \quad t \in \mathbb{R},$$

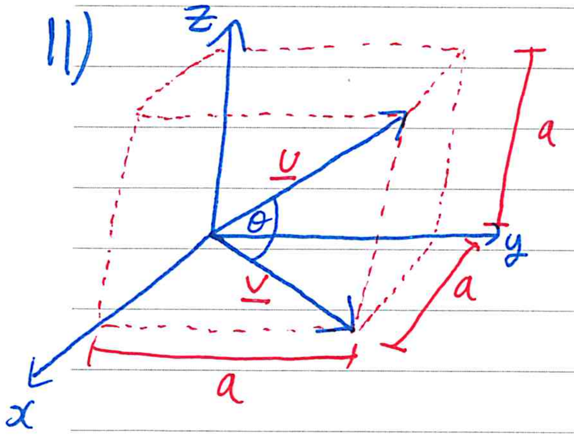
b) $(2,-1,1)\cdot(v_1, v_2, v_3) = 0$

$$2v_1 - v_2 + v_3 = 0 \implies v_1 = \tfrac{1}{2}v_2 - \tfrac{1}{2}v_3$$

$$\therefore \underline{v} = \left(\tfrac{1}{2}(r-t), r, t\right), \quad r, t \in \mathbb{R}$$

or $\underline{v} = (r-t, 2r, 2t)$ (orthogonal to $\underline{u}$)

11)



$\underline{u}$ is cube diagonal

$\underline{v}$ is side diagonal

$\underline{u} = (a, a, a)$

$\underline{v} = (a, a, 0)$

$$\cos\theta = \frac{(a,a,a)\cdot(a,a,0)}{\sqrt{3a^2}\sqrt{2a^2}} = \frac{2a^2}{\sqrt{6a^4}} = \frac{2}{\sqrt{6}}$$

$$\therefore \theta = 0.62^c = 35.26^\circ$$

12) For any 2 vectors, $\underline{a}, \underline{b}$, the dot product is defined to be,

$$\underline{a}\cdot\underline{b} = \sum_{i=1}^{n} a_i b_i$$

for $\underline{a} = (a_1, a_2, \dots, a_n)$, $\underline{b} = (b_1, b_2, \dots, b_n)$,

We have,

$$(\underline{u}+\underline{v})\cdot\underline{w} = \sum_{i=1}^{n}(u_i+v_i)w_i$$

$$= \sum_{i=1}^{n} u_i w_i + v_i w_i$$

$$= \sum_{i=1}^{n} u_i w_i + \sum_{i=1}^{n} v_i w_i = \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w} \quad \square$$

13) $\|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) + (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})$

$$= \underline{u}\cdot(\underline{u}+\underline{v}) + \underline{v}\cdot(\underline{u}+\underline{v}) + \underline{u}\cdot(\underline{u}-\underline{v}) - \underline{v}\cdot(\underline{u}-\underline{v})$$

$$= \underline{u}\cdot\underline{u} + \cancel{\underline{u}\cdot\underline{v}} + \cancel{\underline{v}\cdot\underline{u}} + \underline{v}\cdot\underline{v} + \underline{u}\cdot\underline{u} - \cancel{\underline{u}\cdot\underline{v}} - \cancel{\underline{v}\cdot\underline{u}} + \underline{v}\cdot\underline{v}$$

$$= 2\underline{u}\cdot\underline{u} + 2\underline{v}\cdot\underline{v} = 2\|\underline{u}\|^2 + 2\|\underline{v}\|^2$$

$\square$

SUBJECT: NO: DATE: / /

14) a) $\langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v}$ (Euclidean)

$= 15 - 48 = -33$

$\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\|\underline{v}\| = \sqrt{25 + 144} = \sqrt{169} = 13$

$d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\| = \|(-2, 16)\| = \sqrt{4 + 256} = \sqrt{260}$
$= 2\sqrt{65}$

b) $\langle \underline{u}, \underline{v} \rangle = 3u_1 v_1 + u_2 v_2$

$= 3(-4)(0) + (3)(5) = 15$

$\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle} = \sqrt{3(-4)(-4) + (3)(3)} = \sqrt{57}$

$\|\underline{v}\| = \sqrt{3(0)(0) + (5)(5)} = 5$

$d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\| = \|(-4, -2)\| = \sqrt{3(-4)(-4) + (-2)(-2)}$

$= 2\sqrt{13}$

c) $\langle \underline{u}, \underline{v} \rangle = 2u_1 v_1 + 3u_2 v_2 + u_3 v_3$

$= 2(8)(8) + 3(0)(3) + (-8)(16) = 0$

$\|\underline{u}\| = \sqrt{2(8^2) + 3(0^2) + (-8)^2} = 8\sqrt{3}$

$\|\underline{v}\| = \sqrt{2(8^2) + 3(3^2) + 16^2} = \sqrt{411}$

$d(\underline{u}, \underline{v}) = \|(0, -3, -24)\| = \sqrt{2(0^2) + 3(-3)^2 + (-24)^2} = \sqrt{603} = 3\sqrt{67}$

Double A

SUBJECT: NO: DATE: / /

15) a) $\langle x^2, x^2+1 \rangle = \int_{-1}^{1} x^2(x^2+1)\,dx = \int_{-1}^{1} x^4 + x^2\,dx = \left[\frac{x^5}{5} + \frac{x^3}{3}\right]_{-1}^{1}$

$= \frac{16}{15}$

$\|f\| = \sqrt{\langle x^2, x^2 \rangle}$

$\|f\|^2 = \langle x^2, x^2 \rangle = \int_{-1}^{1} x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$

$\therefore \|f\| = \sqrt{\frac{2}{5}}$

$\|g\|^2 = \int_{-1}^{1} (x^2+1)^2\,dx = \int_{-1}^{1} x^4 + 2x^2 + 1\,dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_{-1}^{1}$

$= \frac{56}{15}$

$\therefore \|g\| = \sqrt{\frac{56}{15}}$

$d(f,g) = \|f - g\|$

$\|f-g\|^2 = \int_{-1}^{1} [x^2 - (x^2+1)]^2\,dx = \int_{-1}^{1} (-1)^2\,dx = [x]_{-1}^{1} = 2$

$\therefore \|f-g\| = \sqrt{2}$

SUBJECT: NO: DATE: / /

b) $\langle x, e^x \rangle = \int_{-1}^{1} x e^x dx = [x e^x]_{-1}^{1} - \int_{-1}^{1} e^x dx$

$= [e^x(x-1)]_{-1}^{1} = 2e^{-1}$

$\|x\|^2 = \int_{-1}^{1} x^2 dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{2}{3}$ $\therefore \|x\| = \sqrt{\frac{2}{3}}$

$\|e^x\|^2 = \int_{-1}^{1} e^{2x} dx = [\frac{1}{2} e^{2x}]_{-1}^{1} = \frac{1}{2}(e^2 - e^{-2})$

$\therefore \|e^x\| = \sqrt{\frac{1}{2}(e^2 - e^{-2})}$

$d(x, e^x) = \|x - e^x\|$

$\|x - e^x\|^2 = \int_{-1}^{1} (x - e^x)^2 dx = \int_{-1}^{1} x^2 - 2xe^x + e^{2x} dx$

$= \frac{2}{3} - 2(2e^{-1}) + \frac{1}{2}(e^2 - e^{-2})$

$\therefore \|x - e^x\| = \sqrt{\frac{2}{3} - \frac{4}{e} + \frac{e^2 - e^{-2}}{2}}$

16) a) $\left\langle \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \right\rangle = 2(-1)(0)+(3)(-2)+(4)(1)+2(-2)(1)$

$= -6$

$\left\| \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix} \right\| = \sqrt{2(-1)^2+3^2+4^2+2(-2)^2} = \sqrt{35}$

$\left\| \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \right\| = \sqrt{2(0^2)+(-2)^2+1^2+2(1^2)} = \sqrt{7}$

$d(\underline{\underline{A}}, \underline{\underline{B}}) = \left\| \begin{pmatrix} -1 & 5 \\ 3 & -3 \end{pmatrix} \right\| = \sqrt{2(-1)^2+5^2+3^2+2(-3)^2} = \sqrt{54} = 3\sqrt{6}$

b) $\left\langle \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \right\rangle = 2(1)(0)+(-1)(1)+(2)(-2)+2(4)(0) = -5$

$\left\| \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \right\| = \sqrt{2(1^2)+(-1)^2+2^2+2(4^2)} = \sqrt{39}$

$\left\| \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \right\| = \sqrt{2(0^2)+1^2+(-2)^2+2(0^2)} = \sqrt{5}$

$d(\underline{\underline{A}}, \underline{\underline{B}}) = \left\| \begin{pmatrix} 1 & -2 \\ 4 & 4 \end{pmatrix} \right\| = \sqrt{2(1^2)+(-2)^2+4^2+2(4^2)} = \sqrt{54}$

$= 3\sqrt{6}$

17) a) $\langle 1-x+3x^2, x-x^2 \rangle = (1)(0)+(-1)(1)+(3)(-1) = -4$

$\|1-x+3x^2\| = \sqrt{1^2+(-1)^2+3^2} = \sqrt{11}$

$\|x-x^2\| = \sqrt{1+(-1)^2} = \sqrt{2}$

$d(p,q) = \|(1-x+3x^2)-(x-x^2)\| = \|1-2x+4x^2\|$

$= \sqrt{1+4+16} = \sqrt{21}$

b) $\langle 1+x^2, 1-x^2 \rangle = 1+0-1 = 0$ (orthogonal)

$\|1+x^2\| = \sqrt{2}$, $\|1-x^2\| = \sqrt{2}$

$d(p,q) = \|(1+x^2)-(1-x^2)\| = \|2x^2\| = 2$

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18) Must satisfy 1 commutivity, 2 scalor 3 & vector distributivity, and 4 self inner product ≥ 0 with $\underline{0} \Leftrightarrow 0$,

1 $\langle \underline{\underline{A}}, \underline{\underline{B}} \rangle = 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22}$

$= 2b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + 2b_{22}a_{22} = \langle \underline{\underline{B}}, \underline{\underline{A}} \rangle$

2 $c\langle \underline{\underline{A}}, \underline{\underline{B}} \rangle = c[2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22}]$

$= 2(ca_{11})b_{11} + (ca_{12})b_{12} + (ca_{21})b_{21} + 2(ca_{22})b_{22}$

$= \langle c\underline{\underline{A}}, \underline{\underline{B}} \rangle$

SUBJECT: NO: DATE: / /

(3) $\langle \underline{\underline{A}}, \underline{\underline{B}} + \underline{\underline{C}} \rangle = 2a_{11}(b_{11}+c_{11}) + a_{12}(b_{12}+c_{12}) + a_{21}(b_{21}+c_{21})$
$+2a_{22}(b_{22}+c_{22})$

$= (2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22})$
$+ (2a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + 2a_{22}c_{22})$

$= \langle \underline{\underline{A}}, \underline{\underline{B}} \rangle + \langle \underline{\underline{A}}, \underline{\underline{C}} \rangle$

(4) $\langle \underline{\underline{A}}, \underline{\underline{A}} \rangle = 2a_{11}^2 + a_{12}^2 + a_{21}^2 + 2a_{22}^2 \geqslant 0$

since all terms are square,

Also solving equal to 0 for any term results in the squareroot of a negative number unless

$a_{11} = a_{12} = a_{21} = a_{22} = 0$

$\therefore \langle \underline{\underline{A}}, \underline{\underline{A}} \rangle = 0 \implies \underline{\underline{A}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

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19) a) $\cos\theta = \dfrac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\|\|\underline{v}\|} = \dfrac{3(-4)(0)+(3)(5)}{\sqrt{3(-4)^2+3^2}\sqrt{3(0^2)+5^2}} = \dfrac{15}{\sqrt{57}(5)}$

$= \dfrac{3}{\sqrt{57}}$

$\therefore \theta = 1.16^c = 66.59^\circ$

SUBJECT: .... NO: .... DATE: / /

b) $\cos\theta = \dfrac{2(1)(2)+(1)(-2)+(1)(2)}{\sqrt{2(1^2)+1^2+1^2}\sqrt{2(2^2)+(-2)^2+2^2}} = \dfrac{4}{\sqrt{4}\sqrt{16}} = \dfrac{1}{2}$

$\therefore \theta = \dfrac{\pi}{3}^c = 60^\circ$

c) $\cos\theta = \dfrac{1-1+1}{\sqrt{3}\sqrt{3}} = \dfrac{1}{3}$ $\therefore \theta = 1.23^c = 70.53^\circ$

d) $\cos\theta = \dfrac{\int_{-1}^{1} x^3\,dx}{\left[\int_{-1}^{1} x^2\,dx\right]^{\frac{1}{2}}\left[\int_{-1}^{1} x^4\,dx\right]^{\frac{1}{2}}} = \dfrac{\left[\frac{x^4}{4}\right]_{-1}^{1}}{\left[\left[\frac{x^3}{3}\right]_{-1}^{1}\right]^{\frac{1}{2}}\left[\left[\frac{x^5}{5}\right]_{-1}^{1}\right]^{\frac{1}{2}}}$

$= 0$ $\therefore \theta = \dfrac{\pi}{2}^c = 90^\circ$

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20) Cauchy-Schwarz; $|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\|\,\|\underline{v}\|$ ①

Triangle inequality; $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ ②

a) $\langle \underline{u}, \underline{v} \rangle = -5+0+4 = -1$

$\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle} = \sqrt{17}$

$\|\underline{v}\| = \sqrt{42}$, $\|\underline{u}+\underline{v}\| = \sqrt{\langle(-4,4,5),(-4,4,5)\rangle}$

$= \sqrt{57}$

① $|\langle \underline{u}, \underline{v} \rangle| = 1 \leq \sqrt{17}\sqrt{42} = 26.72 = \|\underline{u}\|\,\|\underline{v}\|$

② $\|\underline{u}+\underline{v}\| = \underset{\substack{\uparrow \\ 7.55}}{\sqrt{57}} \leq \sqrt{17}+\sqrt{42} = 10.60 = \|\underline{u}\| + \|\underline{v}\|$

b) $\langle p, q \rangle = 0 \; , \; \|p\| = 2, \quad \|q\| = \sqrt{10}$

$\|p+q\| = \| 2x + 3x^2 + 1 \| = \sqrt{14}$

① $|\langle p, q \rangle| = 0 \leq 2\sqrt{10} = \|p\| \, \|q\|$

② $\|p+q\| = \sqrt{14} = 3{,}74 \leq 2 + \sqrt{10} = 5{,}16 = \|p\| + \|q\|$

c) $\langle \underline{\underline{A}}, \underline{\underline{B}} \rangle = 0 + 3 + 8 + 6 = 17$

$\|\underline{\underline{A}}\| = \sqrt{15} \; , \quad \|\underline{\underline{B}}\| = \sqrt{53}$

$$\|\underline{\underline{A}} + \underline{\underline{B}}\| = \left\| \begin{pmatrix} -3 & 4 \\ 6 & 4 \end{pmatrix} \right\| = \sqrt{18 + 16 + 36 + 32} = \sqrt{102}$$

① $|\langle \underline{\underline{A}}, \underline{\underline{B}} \rangle| = 17 \leq \sqrt{15}\sqrt{53} = 28{,}20 = \|\underline{\underline{A}}\| \, \|\underline{\underline{B}}\|$

② $\|\underline{\underline{A}} + \underline{\underline{B}}\| = \sqrt{102} = 10{,}10 \leq \sqrt{15} + \sqrt{53} = 11{,}15 = \|\underline{\underline{A}}\| + \|\underline{\underline{B}}\|$

d) $$\langle f, g \rangle = \int_{-\pi}^{\pi} \sin x \cos x \, dx = \left[\frac{1}{2} \sin^2 x \right]_{-\pi}^{\pi} = 0$$

$$\|f\|^2 = \int_{-\pi}^{\pi} \sin^2 x \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2x) \, dx = \left[\frac{x}{2} - \frac{1}{4} \sin 2x \right]_{-\pi}^{\pi} = \pi$$

$$\|g\|^2 = \int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos 2x) \, dx = \left[\frac{x}{2} + \frac{1}{4} \sin 2x \right]_{-\pi}^{\pi} = \pi$$

$$\|f+g\|^2 = \int_{-\pi}^{\pi} (\sin x + \cos x)^2 \, dx = \int_{-\pi}^{\pi} \sin^2 x + \cos^2 x + 2 \sin x \cos x \, dx$$

SUBJECT: NO: DATE: / /

$= \pi + \pi + \frac{1}{2}0 = 2\pi$

① $|\langle f, g\rangle| = 0 \leq \sqrt{\pi}\sqrt{\pi} = \|f\|\|g\|$

② $\|f+g\| = \sqrt{2\pi} \leq \sqrt{\pi} + \sqrt{\pi} = \|f\| + \|g\|$

21) a) $\langle \cos x, \sin x\rangle = \int_{-\pi}^{\pi} \cos x \sin x \, dx = \left[\frac{1}{2}\sin^2 x\right]_{-\pi}^{\pi} = 0$

$\therefore$ orthogonal.

b) $\langle x, \frac{1}{2}(5x^3 - 3x)\rangle = \int_{-1}^{1} \frac{5x^4}{2} - \frac{3x^2}{2} \, dx = \left[\frac{x^5}{2} - \frac{x^3}{2}\right]_{-1}^{1}$

$= \left(\frac{1}{2} - \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2}\right) = 0$

$\therefore$ orthogonal.

22) a) $\langle \underline{u}, \underline{v}\rangle = \underline{u} \cdot \underline{v}$

$\text{proj}_{\underline{v}}\underline{u} = \frac{\langle \underline{u}, \underline{v}\rangle}{\|\underline{v}\|^2}\underline{v} = \frac{(0-3-2)}{0+1+1}(0,-1,1) = \frac{-5}{2}(0,-1,1)$

b) $\text{proj}_{\underline{v}}\underline{u} = \frac{0+1+6-12}{1+1+4+4}(-1,1,2,2) = \frac{-5}{10}(-1,1,2,2)$

$= \left(\frac{1}{2}, -\frac{1}{2}, -1, -1\right)$

c) $\langle x, e^x\rangle = \int_0^1 x e^x \, dx = \left[e^x(x-1)\right]_0^1 = 1$

$\|e^x\|^2 = \int_0^1 e^{2x} \, dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1)$

SUBJECT: NO: DATE: / /

$$\text{proj}_g f = \frac{1}{\frac{1}{2}(e^2-1)} e^x = \frac{2e^x}{e^2-1}$$

d) $\langle \cos x, \sin x \rangle = 0$

$$\text{proj}_g f = 0$$

e) $$\langle x, \sin 2x \rangle = \int_{-\pi}^{\pi} x \sin 2x \, dx = \left[-x \frac{1}{2} \cos 2x\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{2} \cos 2x \, dx$$

$$= \left[-\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x\right]_{-\pi}^{\pi} = -\pi$$

$$\langle \sin 2x, \sin 2x \rangle = \int_{-\pi}^{\pi} \sin^2 2x \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 4x) \, dx$$

$$= \left[\frac{x}{2} - \frac{\sin 4x}{8}\right]_{-\pi}^{\pi} = \pi$$

$$\text{proj}_g f = \frac{-\pi}{\pi} \sin 2x = -\sin 2x$$

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23) a) Using the 4 axioms for an inner product:

$\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{w}, \underline{u} + \underline{v} \rangle$ axiom 1

$= \langle \underline{w}, \underline{u} \rangle + \langle \underline{w}, \underline{v} \rangle$ axiom 2

$= \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$ axiom 1

SUBJECT: NO: DATE: / /

b) $\langle \underline{u}, c\underline{v} \rangle = \langle c\underline{v}, \underline{u} \rangle$ axiom 1

$= c\langle \underline{v}, \underline{u} \rangle$ axiom 3

$= c\langle \underline{u}, \underline{v} \rangle$ axiom 1

$= \langle c\underline{u}, \underline{v} \rangle$ axiom 3

c) Q is the set of vectors orthogonal to every vector in W,

Closed under scalar multiplication & addition $\Rightarrow$ subspace.
(2) (1)

(1) $\langle \underline{v}_1, \underline{w} \rangle = 0$, $\langle \underline{v}_2, \underline{w} \rangle = 0$

$\langle \underline{v}_1 + \underline{v}_2, \underline{w} \rangle = \langle \underline{v}_1, \underline{w} \rangle + \langle \underline{v}_2, \underline{w} \rangle = 0$

$\therefore (\underline{v}_1 + \underline{v}_2) \in Q$

(2) $\langle c\underline{v}, \underline{w} \rangle = c\langle \underline{v}, \underline{w} \rangle$ 3rd inner product axiom

$= c(0) = 0$

$\therefore c\underline{v} \in Q$