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1) a) $(2,-4)\cdot(2,1)=0$ $\therefore$ orthogonal

$\|(2,-4)\|=\sqrt{4+16}=\sqrt{20}\neq 1$ $\therefore$ not orthonormal

b) $(-4,6)\cdot(5,0)=-20\neq 0$ $\therefore$ not orthogonal

c) $$\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)$$

$$=\frac{-\sqrt{12}}{2\sqrt{6}}+0+\frac{\sqrt{12}}{2\sqrt{6}}=0$$

$$\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3}\right)$$

$$=\frac{\sqrt{6}}{6}+0-\frac{\sqrt{6}}{6}=0$$

$$\left(\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\cdot\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3}\right)$$

$$=\frac{-\sqrt{18}}{18}+\frac{\sqrt{18}}{9}-\frac{\sqrt{18}}{18}=0$$

$\therefore$ orthogonal

$$\left\|\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\right\|=\sqrt{\frac{2}{4}+0+\frac{2}{4}}=1$$

$$\left\|\left(\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\right\|=\sqrt{\frac{6}{36}+\frac{6}{9}+\frac{6}{36}}=1$$

$$\left\|\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3}\right)\right\|=\sqrt{\frac{3}{9}+\frac{3}{9}+\frac{3}{9}}=1$$ $\therefore$ orthonormal

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2) Remember the standard basis for $\mathbb{R}^n$ has the identity matrix as its transition matrix so:

$$\underline{\underline{P}}\,[\underline{x}]_B = \underline{\underline{Q}}\,[\underline{x}]_{B'} \quad \text{if } B' = S \text{ gives,}$$

$$\underline{\underline{P}}\,[\underline{x}]_B = \underline{\underline{I}}\,[\underline{x}]_S = [\underline{x}]_S$$

$$\therefore \underbrace{[\underline{x}]_B}_{\text{what we want}} = \underline{\underline{P}}^{-1}[\underline{x}]_S$$

Since $\underline{\underline{P}}$ is orthogonal $\underline{\underline{P}}^{-1} = \underline{\underline{P}}^T$. Considering a 2x2,

$$\underline{\underline{P}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{for a basis, } \{(a,c),(b,d)\}$$

$$\underline{\underline{P}}^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow [\underline{x}]_B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}[\underline{x}]_S = \begin{pmatrix} (a,c)\cdot[\underline{x}]_S \\ (b,d)\cdot[\underline{x}]_S \end{pmatrix}$$

The coordinates, $[\underline{x}]_B$, are just the dot product of the basis vectors in B with $[\underline{x}]_S$,

a) $$[\underline{x}]_B = \begin{pmatrix} \left(\frac{-2\sqrt{13}}{13}, \frac{3\sqrt{13}}{3}\right)\cdot(1,2) \\ \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13}\right)\cdot(1,2) \end{pmatrix} = \begin{pmatrix} \frac{4\sqrt{13}}{13} \\ \frac{7\sqrt{13}}{13} \end{pmatrix}$$

b) $$[x]_B = \begin{pmatrix} \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10}\right) \cdot (2,-2,1) \\ (0,1,0) \cdot (2,-2,1) \\ \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10}\right) \cdot (2,-2,1) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{10}}{2} \\ -2 \\ -\frac{\sqrt{10}}{2} \end{pmatrix}$$

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3) a) $$B = \{\underbrace{(3,4)}_{\underline{v}_1}, \underbrace{(1,0)}_{\underline{v}_2}\}$$

Let $\underline{w}_1 = \underline{v}_1 = (3,4)$

"Euclidean"-dot product

$$\underline{w}_2 = \underline{v}_2 - \frac{\langle \underline{v}_2, \underline{w}_1 \rangle}{\langle \underline{w}_1, \underline{w}_1 \rangle} \underline{w}_1$$

$$= (1,0) - \frac{(1,0)\cdot(3,4)}{\|(3,4)\|^2}(3,4)$$

$$= (1,0) - \frac{3}{25}(3,4) = \left(\frac{16}{25}, \frac{-12}{25}\right)$$

$$\|\underline{w}_1\| = 5, \quad \|\underline{w}_2\| = \sqrt{\frac{256}{625} + \frac{144}{625}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$\therefore$ orthonormal basis is $\left\{ \frac{\underline{w}_1}{\|\underline{w}_1\|}, \frac{\underline{w}_2}{\|\underline{w}_2\|} \right\}$

$$= \left\{ \left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, -\frac{3}{5}\right) \right\}$$

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b) $B = \{ \underset{v_1}{(4,-3,0)}, \underset{v_2}{(1,2,0)}, \underset{v_3}{(0,0,4)} \}$

Let $w_1 = v_1 = (4,-3,0)$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = (1,2,0) - \frac{(1,2,0)\cdot(4,-3,0)}{\|(4,-3,0)\|^2}(4,-3,0)$$

$$= (1,2,0) - \frac{-2}{25}(4,-3,0) = \left(\frac{33}{25}, \frac{44}{25}, 0\right)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= (0,0,4) - \frac{(0,0,4)\cdot(4,-3,0)}{\|(4,-3,0)\|^2}(4,-3,0) - \frac{(0,0,4)\cdot\left(\frac{33}{25}, \frac{44}{25}, 0\right)}{\left\|\left(\frac{33}{25}, \frac{44}{25}, 0\right)\right\|}\left(\frac{33}{25}, \frac{44}{25}, 0\right)$$

$$= (0,0,4) - 0 - 0 = (0,0,4)$$

$$\|w_1\| = 5, \quad \|w_2\| = \sqrt{\left(\frac{33}{25}\right)^2 + \left(\frac{44}{25}\right)^2} = \sqrt{\frac{121}{25}} = \frac{11}{5}$$

$\|w_3\| = 4$ $\therefore$ orthonormal basis is,

$$\left\{ \left(\frac{4}{5}, \frac{-3}{5}, 0\right), \left(\frac{3}{5}, \frac{4}{5}, 0\right), (0,0,1) \right\}$$

c) $B = \{\underset{\underline{v}_1}{(3,4,0,0)}, \underset{\underline{v}_2}{(-1,1,0,0)}, \underset{\underline{v}_3}{(2,1,0,-1)}, \underset{\underline{v}_4}{(0,1,1,0)}\}$

Let $\underline{w}_1 = \underline{v}_1 = (3,4,0,0)$

$$\underline{w}_2 = (-1,1,0,0) - \frac{(-1,1,0,0)\cdot(3,4,0,0)}{\|(3,4,0,0)\|^2}(3,4,0,0)$$

$$= \left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)$$

$$\underline{w}_3 = (2,1,0,-1) - \frac{(2,1,0,-1)\cdot\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)}{\left\|\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)\right\|^2}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)$$

$$- \frac{(2,1,0,-1)\cdot(3,4,0,0)}{\|(3,4,0,0)\|^2}(3,4,0,0)$$

$$= (0,0,0,-1)$$

$$\underline{w}_4 = (0,1,1,0) - \frac{(0,1,1,0)\cdot(0,0,0,-1)}{\|(0,0,0,-1)\|^2}(0,0,0,-1)$$

$$- \frac{(0,1,1,0)\cdot\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)}{\left\|\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)\right\|^2}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)$$

$$- \frac{(0,1,1,0)\cdot(3,4,0,0)}{\|(3,4,0,0)\|^2}(3,4,0,0)$$

$$= (0,0,1,0)$$

$\|\underline{w}_1\| = 5$, $\|\underline{w}_2\| = \frac{7}{5}$, $\|\underline{w}_3\| = 1$, $\|\underline{w}_4\| = 1$

So orthonormal basis is,

$$\left\{\left(\tfrac{3}{5}, \tfrac{4}{5}, 0, 0\right), \left(-\tfrac{4}{5}, \tfrac{3}{5}, 0, 0\right), (0,0,0,-1), (0,0,1,0)\right\}$$

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4) Find a basis for Nul $\underline{\underline{A}}$ then apply Gram-Schmidt,

$$\begin{pmatrix} 2 & 1 & -6 & 2 \\ 1 & 2 & -3 & 4 \\ 1 & 1 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -6 & 2 \\ 0 & 3/2 & 0 & 3 \\ 0 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{aligned} x_1 &= 3x_3 \\ x_2 &= -2x_4 \end{aligned}$$

$$\therefore \underline{x} = x_3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

A basis is $\{(3,0,1,0), (0,-2,0,1)\}$

Make orthonormal;

Let $\underline{w}_1 = (3,0,1,0)$

$$\underline{w}_2 = (0,-2,0,1) - \frac{(0,-2,0,1)\cdot(3,0,1,0)}{\|(3,0,1,0)\|^2}(3,0,1,0)$$

$$= (0,-2,0,1)$$

$\|\underline{w}_1\| = \sqrt{10}, \quad \|\underline{w}_2\| = \sqrt{5}$

$\therefore$ orthonormal basis is,

$$\left\{ \left(\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}, 0\right), \left(0, \frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right) \right\}$$

Alternatively, note that $(3,0,1,0)\cdot(0,-2,0,1) = 0$

$\therefore$ orthogonal already, so can just normalise these.

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5) $B = \{(2,-1), (-2,10)\}$, $\langle \underline{u}, \underline{v} \rangle = 2u_1v_1 + u_2v_2$

Let $\underline{w}_1 = (2,-1)$

$$\underline{w}_2 = (-2,10) - \frac{\langle (-3,10), (2,-1) \rangle}{\langle (2,-1), (2,-1) \rangle}(2,-1)$$

$$= (-2,10) - \frac{2(-2)(2) + (10)(-1)}{2(2)(2) + (-1)(-1)}(2,-1)$$

$$= (2,8)$$

$\|\underline{w}_1\| = \sqrt{\underline{w}_1 \cdot \underline{w}_1} = \sqrt{2(2^2) + (-1)^2} = 3$, $\|\underline{w}_2\| = \sqrt{2(2^2) + 8^2}$

$= 6\sqrt{2}$

$\therefore$ orthonormal basis is,

$$\left\{ \left(\frac{2}{3}, \frac{-1}{3}\right), \left(\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right) \right\}$$

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6) Need bases for $Nul(\underline{\underline{A}})$, $Nul(\underline{\underline{A}}^T)$, $Col(\underline{\underline{A}})$, $Col(\underline{\underline{A}}^T)$,

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{A}}^T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\underline{\underline{A}} \Rightarrow x_1 = \frac{3}{2} x_3, \quad x_2 = -\frac{x_3}{2}$

$$\underline{x} = x_3 \begin{pmatrix} 3/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$\underline{\underline{A}}^T \Rightarrow x_1 = -x_3, \quad x_2 = -x_3$

$$\underline{x} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Basis for $Nul(\underline{\underline{A}})$: $\{(3, -1, 2)\}$

Basis for $Nul(\underline{\underline{A}}^T)$: $\{(-1, -1, 1)\}$

Basis for $Col(\underline{\underline{A}})$: $\{(1, 0, 1), (1, 2, 3)\}$

Basis for $Col(\underline{\underline{A}}^T)$: $\{(1, 1, -1), (0, 2, 1)\}$

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7) a) $(2,1,-1)\cdot(-1,2,0)=0$ ✓

$(0,1,1)\cdot(-1,2,0)=2\neq 0$ ✗

Not orthogonal sets.

b) $(1,1,1,1)\cdot(-1,1,-1,1)=0$ ✓

$(1,1,1,1)\cdot(0,2,-2,0)=0$ ✓

Orthogonal sets.

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8) a) $S=\text{Span}\{(1,2,0,0),(0,1,0,1)\}$

$S^{\perp}=\{\underline{v}=(v_1,v_2,v_3,v_4)\in\mathbb{R}^4;\ (1,2,0,0)\cdot\underline{v}=(0,1,0,1)\cdot\underline{v}=0\}$

$$\begin{pmatrix}1&2&0&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}=\underline{0}$$

$$\Longrightarrow\begin{pmatrix}1&2&0&0\\0&1&0&1\end{pmatrix}\sim\begin{pmatrix}1&0&0&-2\\0&1&0&1\end{pmatrix}\Longrightarrow \begin{aligned}v_1&=2v_4\\v_2&=-v_4\end{aligned}$$

v_4, v_3 free

$$\underline{v}=v_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+v_4\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$$

$\therefore\ S^{\perp}=\text{Span}\{(2,-1,0,1),(0,0,1,0)\}$

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b) $S = \text{Span}\{(1,0,0), (0,0,1)\}$ ← x and z axes

$S^{\perp} = \text{Span}\{(0,1,0)\}$ ← y axis

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9) a) $S = \text{Span}\{(0,0,-1,1), (0,1,1,1)\}$, $\underline{v} = (1,0,1,1)$

$\text{proj}_S \underline{v} = \sum_{j=1}^{p} (\underline{v} \cdot \underline{u}_j)\underline{u}_j$, where $\{\underline{u}_j\}_1^p$ is an orthonormal basis.

Note that $(0,0,-1,1) \cdot (0,1,1,1) = 0$ ∴ orthogonal.

$\|(0,0,-1,1)\| = \sqrt{2}$, $\|(0,1,1,1)\| = \sqrt{3}$

Orthonormal basis is $\left\{\left(0,0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \left(0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\}$

$$\therefore \text{proj}_S \underline{v} = \left[(1,0,1,1) \cdot \left(0,0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right]\left(0,0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) + \left[(1,0,1,1) \cdot \left(0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right]\left(0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$= \left(0,\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$$

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b) $S = \text{Span}\{(1,0,1), (0,1,1)\}$, $\underline{v} = (2,3,4)$

Use Gram-Schmidt to get orthonormal basis:

Let $\underline{w}_1 = (1,0,1)$, $\|\underline{w}_1\| = \sqrt{2}$

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$$\underline{w}_2 = (0,1,1) - \frac{(0,1,1)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1)$$

$$= \left(-\frac{1}{2}, 1, \frac{1}{2}\right) \quad , \quad \|\underline{w}_2\| = \sqrt{\frac{3}{2}}$$

Orthonormal basis is,

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{\sqrt{6}}{6}, \sqrt{\frac{3}{2}}, \frac{\sqrt{6}}{6}\right)\right\}$$

$$\text{proj}_S \underline{v} = \left[(2,3,4)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$+ \left[(2,3,4)\cdot\left(-\frac{\sqrt{6}}{6}, \sqrt{\frac{3}{2}}, \frac{\sqrt{6}}{6}\right)\right]\left(-\frac{\sqrt{6}}{6}, \sqrt{\frac{3}{2}}, \frac{\sqrt{6}}{6}\right)$$

$$= \left(\frac{5}{3}, \frac{8}{3}, \frac{13}{3}\right)$$

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c) $S = \text{Col}(\underline{\underline{A}})$, $\underline{v} = (2,-2,1)$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{Col}(\underline{\underline{A}}) = \text{Span}\{(1,0,1), (2,1,1)\}$$

Gram-Schmidt; $\underline{w}_1 = (1,0,1)$, $\|\underline{w}_1\| = \sqrt{2}$

$$\underline{w}_2 = (2,1,1) - \frac{(2,1,1)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1)$$

$$= \left(\frac{1}{2}, 1, -\frac{1}{2}\right) \quad , \quad \|\underline{w}_2\| = \sqrt{\frac{3}{2}}$$

Orthonormal basis is,

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right)\right.$$

$$\text{proj}_S \underline{v} = \left[(2,-2,1)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$
$$+ \left[(2,-2,1)\cdot\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right)\right]\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right)$$

$$= (1, -1, 2)$$

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10) Solve $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}^T \underline{b}$,

a) $$\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}$$

$$\underline{\underline{A}}^T \underline{b} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 6 & 5 & 1 \\ 5 & 6 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & 5 & 1 \\ 0 & \frac{11}{6} & -\frac{11}{6} \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right)$$

$$\therefore \underline{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) $\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

$\underline{\underline{A}}^T \underline{b} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 2 & 2 & 4 \\ 2 & 3 & 2 & 0 \\ 2 & 2 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & 2 & 4 \\ 0 & 5/3 & 2/3 & -8/3 \\ 0 & 2/3 & 5/3 & 1/3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & 2 & 4 \\ 0 & 5/3 & 2/3 & -8/3 \\ 0 & 0 & 7/5 & 7/5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \underline{x} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

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11) a) $y = ax + b$

$$\begin{aligned} 1 &= -a + b \\ 0 &= a + b \\ -3 &= 3a + b \end{aligned} \Rightarrow \begin{array}{c} \left(\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -3 \end{array}\right) \\ \underline{\underline{A}} \qquad \underline{b} \end{array}$$

$\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 3 \\ 3 & 3 \end{pmatrix}$

$\underline{\underline{A}}^T \underline{b} = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \end{pmatrix}$

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$$\left(\begin{array}{cc|c} 11 & 3 & -10 \\ 3 & 3 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 11 & 3 & -10 \\ 0 & \frac{24}{11} & \frac{8}{11} \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{3} \end{array}\right)$$

$$\Rightarrow a = -1, \quad b = \frac{1}{3}$$

Line of best fit: $y = -x + \frac{1}{3}$

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b) $$\begin{aligned} -1 &= -2a + b \\ 0 &= -a + b \\ 0 &= a + b \\ 2 &= 2a + b \end{aligned} \Rightarrow \left(\begin{array}{cc|c} -2 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{array}\right)$$

$\underline{\underline{A}}$ $\quad$ $\underline{b}$

$$\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\underline{\underline{A}}^T \underline{b} = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 10 & 0 & 6 \\ 0 & 4 & 1 \end{array}\right) \Rightarrow a = \frac{3}{5}, \quad b = \frac{1}{4}$$

Line of best fit: $y = \frac{3}{5}x + \frac{1}{4}$