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1) a) $T(v_1, v_2) = (v_1 + v_2,\ v_1 - v_2)$

Image of $\underline{v} = (3,-4)$; $T(3,-4) = (-1, 7)$

Pre-image of $\underline{w} = (3, 19)$;

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 19 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & 16 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & -8 \end{array}\right)$$

Pre-image is $(11, -8)$

b) $T(v_1, v_2, v_3) = (v_2 - v_1,\ v_1 + v_2,\ 2v_1)$

Image of $\underline{v} = (2,3,0)$; $T(2,3,0) = (1, 5, 4)$

Pre-image of $\underline{w} = (-11, -1, 10)$;

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & -11 \\ 1 & 1 & 0 & -1 \\ 2 & 0 & 0 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v_3 \text{ free}$$

Pre-image is $(5, -6, t),\ t \in \mathbb{R}$

c) $T(v_1, v_2, v_3) = (4v_2 - v_1,\ 4v_1 + 5v_2)$

Image of $\underline{v} = (2,-3,-1)$; $T(2,-3,-1) = (-14, -7)$

$$\left(\begin{array}{cc|c} -1 & 4 & 3 \\ 4 & 5 & 9 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -4 & -3 \\ 0 & 21 & 21 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Pre-image is $(1, 1, t),\ t \in \mathbb{R}$

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2) a) $T(x_1+x_2, y_1+y_2) = (x_1+y_1, 1)$

$T(x_1,y_1) + T(x_2,y_2) = (x_1, 1) + (x_2, 1) = (x_1+x_2, 2)$

Not closed under addition $\therefore$ not a linear transformation (LT)

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b) $T(x_1+x_2, y_1+y_2, z_1+z_2) = ((x_1+x_2)+(y_1+y_2), (x_1+x_2)-(y_1+y_2), z_1+z_2)$

$= (x_1+y_1, x_1-y_1, z_1) + (x_2+y_2, x_2-y_2, z_2)$

$= T(x_1,y_1,z_1) + T(x_2,y_2,z_2)$

$T(c(x,y,z)) = T(cx, cy, cz)$

$= (cx+cy, cx-cy, cz) = cT(x,y,z)$

$\therefore$ is a LT.

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c) $T(\underline{\underline{A}}+\underline{\underline{B}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}(\underline{\underline{A}}+\underline{\underline{B}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\underline{\underline{A}} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\underline{\underline{B}}$

$= T(\underline{\underline{A}}) + T(\underline{\underline{B}})$

$T(c\underline{\underline{A}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}c\underline{\underline{A}} = c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\underline{\underline{A}} = cT(\underline{\underline{A}})$

$\therefore$ is a LT.

d) $T(c(x,y)) = T(cx, cy) = (\sqrt{cx}, c^2xy, \sqrt{cy})$

$cT(x,y) = c(\sqrt{x}, xy, \sqrt{y}) = (c\sqrt{x}, cxy, c\sqrt{y})$

Not closed under scalar multiplication

$\therefore$ not a LT.

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3) a) $(0,3,1) = 0(1,0,0) + 3(0,1,0) + 1(0,0,1)$

$\therefore T(0,3,1) = 3T(0,1,0) + T(0,0,1)$

$= 3(1,3,-2) + (0,-2,2) = (3,7,-4)$

b) $(2,-4,1) = 2(0,1,0) - 4(0,1,0) + (0,0,1)$

$\therefore T(2,-1,1) = 2(2,4,-1) - 4(1,3,-2) + (0,-2,2)$

$= (0,-6,8)$

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4) a) $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$

$\Rightarrow (2,1,0) = 0(1,1,1) - (0,-1,2) + 2(1,0,1)$

$\therefore T(2,1,0) = -T(0,-1,2) + 2T(1,0,1)$

$= -(-3,2,-1) + 2(1,1,0) = (5,0,1)$

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b) $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & -1 & 0 & -1 \\ 1 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 2 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 7/2 \end{array}\right)$

$\Rightarrow (2,-1,1) = -\frac{3}{2}(1,1,1) - \frac{1}{2}(0,-1,2) + \frac{7}{2}(1,0,1)$

$\therefore T(2,-1,1) = -\frac{3}{2}(2,0,-1) - \frac{1}{2}(-3,2,-1) + \frac{7}{2}(1,1,0)$

$= (2, \frac{5}{2}, 2)$

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5) $\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore T\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + 3\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 12 & -1 \\ 7 & 4 \end{pmatrix}$

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6) a) $Ker(T) = \{(v_1, v_2, v_3) ; v_1, v_2, v_3 \in \mathbb{R}\}$

(All vectors in $\mathbb{R}^3$ get transformed to $\underline{0}$)

b) $Ker(T) = \{(0,t,0) : t \in \mathbb{R}\}$

c) $\begin{array}{l} x + 2y = 0 \\ y - x = 0 \end{array} \Rightarrow x = y = 0$

$\therefore Ker(T) = \{(0,0)\}$

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7) Kernel is solution to homogeneous equation,
Range is column space,

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ Only trivial solution

$\therefore Ker(T) \Rightarrow \{(0,0)\}$ (basis)

2 linearly independent columns $\therefore Range(T) \Rightarrow \{(1,3),(2,4)\}$ (basis)

Alternatively: $\underline{A}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Range(T) = \{(1,0),(0,1)\}$

b) $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \underline{\ } = x_3 \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

$\therefore Ker(T) = \{(-4t, -2t, t) : t \in \mathbb{R}\}$

Basis is $\{(-4,-2,1)\}$

2 linearly independent columns $\therefore Range(T) \Rightarrow \{(1,0),(-1,1)\}$ (basis)

c) $\begin{pmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -13 \\ 0 & 5 & -5 & 13 \\ 0 & 0 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow x_3$ free

$\underline{x} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$Ker(T) = \{(-t, t, t, 0) : t \in \mathbb{R}\}$, $Range(T) \Rightarrow \left\{ \begin{pmatrix} 1 \\ 3 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ -3 \\ 1 \end{pmatrix} \right\}$ (basis)

Basis is $\{(-1,1,1,0)\}$

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8) a) $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Only trivial solution

$\therefore Ker(T) = \{(0,0)\}$

$nul(T) = dim(Ker(T)) = 0$

$Range(T) = Span\{(-1,1), (1,1)\}$

$Rank(T) = dim(col(\underline{A})) = dim(Range(T)) = 2$

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b) $\begin{pmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 11/4 \\ 0 & 1 & -3/2 \end{pmatrix} \Rightarrow \underline{x} = x_3 \begin{pmatrix} -11/4 \\ 3/2 \\ 1 \end{pmatrix}$

$Ker(T) = Span\{(-\frac{11}{4}, \frac{3}{2}, 1)\}$

$nul(T) = dim(Ker(T)) = 1$

$Range(T) = Span\{(0,4), (-2,0)\}$

$Rank(T) = 2$

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9) $T(p) = \frac{dp}{dx} = \frac{d}{dx}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right] = 0$

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 = 0$

$\Rightarrow a_1 = a_2 = a_3 = a_4 = 0$ to be valid for all x,

a_0 is free.

$\therefore Ker(T) = \{p(x) = a_0 ; a_0 \in \mathbb{R}\}$

(constant polynomials)

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10) $\underline{u} = (u_1, u_2, u_3)$, $\underline{v} = (2, -1, 1)$

$$T(\underline{u}) = \text{proj}_{\underline{v}}\, \underline{u} = \frac{(u_1, u_2, u_3)\cdot(2,-1,1)}{\|(2,-1,1)\|^2}(2,-1,1)$$

$$= \frac{(2u_1 - u_2 + u_3)}{6}(2,-1,1)$$

Find $T(\underline{u}) = \underline{0}$; $2u_1 - u_2 + u_3 = 0$

$$\Rightarrow \underline{u} = u_2\begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix} + u_3\begin{pmatrix}-\frac{1}{2}\\0\\1\end{pmatrix}$$

$\ker(T) \Rightarrow \{(\frac{1}{2}, 1, 0), (-\frac{1}{2}, 0, 1)\}$
(basis)

$\text{null}(T) = 2$, $\text{rank}(T) = 3 - 2 = 1$

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11) a) $T(x,y) = (x+2y, x-2y)$, $\underline{\underline{A}} = \begin{pmatrix}1 & 2\\1 & -2\end{pmatrix}$

b) $T(x,y) = (2x-3y, x-y, y-4x)$

$$\underline{\underline{A}} = \begin{pmatrix}2 & -3\\1 & -1\\-4 & 1\end{pmatrix}$$

c) $T(x,y,z) = (x+y, x-y, z-x)$, $\underline{\underline{A}} = \begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\-1 & 0 & 1\end{pmatrix}$

Double A

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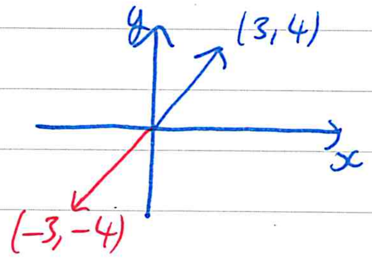
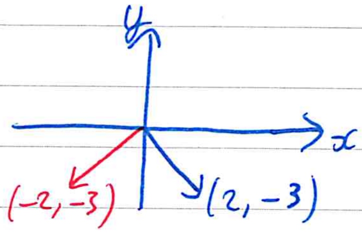
12) a) $\underline{\underline{A}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $T(3,4) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$



Reflection about the line $y = \frac{-3x}{4}$,

b) $\underline{\underline{A}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $T(2,-3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$



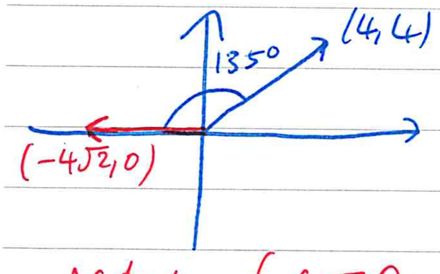
Reflection about y-axis,

c) $\underline{\underline{A}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\underset{(\theta = 135°)}{T(4,4)} = \begin{pmatrix} \cos 135 & -\sin 135 \\ \sin 135 & \cos 135 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$\cos(135) = -\frac{\sqrt{2}}{2}$

$\sin(135) = \frac{\sqrt{2}}{2}$

$= \begin{pmatrix} -2\sqrt{2} - 2\sqrt{2} \\ 2\sqrt{2} - 2\sqrt{2} \end{pmatrix} = \begin{pmatrix} -4\sqrt{2} \\ 0 \end{pmatrix}$



Rotation about origin of 135°.
(Anti-clockwise)

Note: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is known as a rotation matrix.

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13) $\underline{\underline{A}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $T(1,2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$(\theta = 60)$

$\cos 60 = \frac{1}{2}$
$\sin 60 = \frac{\sqrt{3}}{2}$

$$= \begin{pmatrix} \frac{1+2\sqrt{3}}{2} \\ \frac{2-\sqrt{3}}{2} \end{pmatrix}$$

14) xy-plane reflection; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

xz-plane reflection; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

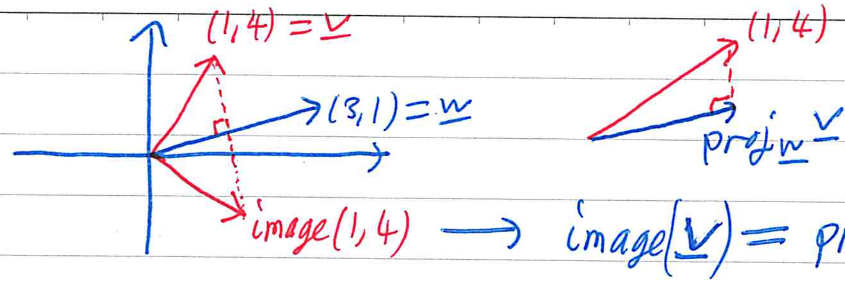
yz-plane reflection; $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

15) a) $T(\underline{v}) = \text{proj}_{\underline{w}}\underline{v} = \frac{(v_1, v_2)\cdot(3,1)}{3^2+1^2}(3,1) = \frac{3v_1+v_2}{10}(3,1)$

$$= \left(\frac{9v_1+3v_2}{10}, \frac{3v_1+v_2}{10}\right)$$

$\underline{\underline{A}} = \begin{pmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$, $T(1,4) = \begin{pmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{21}{10} \\ \frac{7}{10} \end{pmatrix}$

b)



$$\longrightarrow \text{image}(\underline{v}) = \text{proj}_{\underline{w}}\underline{v} - \underline{v} + \text{proj}_{\underline{w}}\underline{v}$$

$$= 2\,\text{proj}_{\underline{w}}\underline{v} - \underline{v}$$

$$T(\underline{v}) = 2\left(\frac{9v_1+3v_2}{10}, \frac{3v_1+v_2}{10}\right) - (v_1, v_2)$$

$$= \left(\frac{4}{5}v_1 + \frac{3v_2}{5}, \frac{3v_1}{5} - \frac{4}{5}v_2\right)$$

$$\underline{\underline{A}} = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}, \quad T(1,4) = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 16/5 \\ -13/5 \end{pmatrix}$$

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16) a) $\underline{\underline{A}}_1 = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}, \quad \underline{\underline{A}}_2 = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$

$$T = T_2 \circ T_1 \Rightarrow \underline{\underline{A}} = \underline{\underline{A}}_2\,\underline{\underline{A}}_1 = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix}$$

b) $\underline{\underline{A}}_1 = \begin{pmatrix} -1 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \underline{\underline{A}}_2 = \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ -2 & 5 \end{pmatrix}$$

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17) a) $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $|\underline{\underline{A}}| = -2 \neq 0$ $\therefore$ invertible,

$$\underline{\underline{A}}^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$\therefore T^{-1}(x,y) = \left(\frac{x}{2} + \frac{y}{2}, \frac{x}{2} - \frac{y}{2}\right)$

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b) $\underline{\underline{A}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $|\underline{\underline{A}}| = 0$ $\therefore$ not invertible

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c) $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $|\underline{\underline{A}}| = 1 \neq 0$ $\therefore$ invertible

$$\underline{\underline{A}}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$ (Adjunct formula)

$\therefore T^{-1}(x,y,z) = (x, y-x, z-y)$

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d) $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$, $|\underline{\underline{A}}| = 0$ $\therefore$ not invertible.

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18) a) $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ Since $T: \mathbb{R}^2 \to \mathbb{R}^3$

Can't use square matrix method,

$\Longrightarrow$ Transform basis vectors of B;

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\Longrightarrow$ Write in terms of basis vectors in B';

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \underline{-1} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \underline{1} \end{array}\right)$$

$$\Longrightarrow \underline{\underline{A}}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}$$

$\Longrightarrow \underline{v} = (5, 4)$ (standard basis)

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ -1 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 9 \end{array}\right) \therefore [\underline{v}]_B = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

Basis, B

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$$[\underline{w}]_{B'} = \underline{A}^*[\underline{v}]_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$$

Check:

$$\underline{A}\,\underline{v} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 4 \end{pmatrix} = \underline{w} \text{ (standard basis)}$$

same

$$[\underline{w}]_{B'} \rightarrow \underline{w}$$

$$\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} \rightarrow 5\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 4 \end{pmatrix}$$

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b) $\underline{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

Since $T: \underbrace{\mathbb{R}^{3} \rightarrow \mathbb{R}^{2}}$

Can't use square matrix method.

$\Longrightarrow$ Transform basis vectors of B:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

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$\Rightarrow$ Write in terms of basis vectors in B';

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \underline{0} \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \underline{-1} \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \underline{-2} \end{array}\right)$$

$$\Rightarrow \underline{\underline{A}}^* = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

$\Rightarrow \underline{v} = (1, 2, -3)$ (standard basis)

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Basis, B

$$\therefore [\underline{v}]_B = \begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$$

$$\underline{\underline{[\underline{w}]_{B'}}} = \underline{\underline{A}}^* [\underline{v}]_B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} 6 \\ -7 \end{pmatrix}}$$

Check! $\underline{\underline{A}}\underline{v} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \underline{w}$ (standard basis)

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$[\underline{w}]_{B'} \longrightarrow \underline{w}$

$$\begin{pmatrix} 6 \\ -7 \end{pmatrix} \longrightarrow 6\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 7\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$ ✓

c) $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}$ $\quad T: \underbrace{\mathbb{R}^3 \longrightarrow \mathbb{R}^3}$

Can use square matrix method.

Since $B' \neq S$ (standard matrix)

we have, $\underline{\underline{A}}^* = \underline{\underline{Q}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}}$,

$\underline{\underline{P}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $\underline{\underline{Q}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $|\underline{\underline{Q}}| = 1$

$\underline{\underline{Q}}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$, $\underline{\underline{A}}^* = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 3 & 4 \\ 0 & 2 & 1 \\ -1 & 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 6 \\ 1 & -1 & -2 \\ -3 & -1 & -3 \end{pmatrix}$$

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$\Rightarrow \underline{v} = (4, -5, 10)$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 0 & 2 & 2 & -5 \\ 1 & 1 & 1 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 1 & -5/2 \\ 0 & 1 & \frac{1}{2} & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 1 & -5/2 \\ 0 & 0 & -\frac{1}{2} & \frac{21}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 25/2 \\ 0 & 1 & 0 & 37/2 \\ 0 & 0 & 1 & -21 \end{array}\right) \quad \therefore \; [\underline{v}]_B = \begin{pmatrix} 25/2 \\ 37/2 \\ -21 \end{pmatrix}$$

$$\underline{\underline{A}}^*[\underline{v}]_B = \begin{pmatrix} 2 & 4 & 6 \\ 1 & -1 & -2 \\ -3 & -1 & -3 \end{pmatrix}\begin{pmatrix} 25/2 \\ 37/2 \\ -21 \end{pmatrix} = \begin{pmatrix} -27 \\ 36 \\ 7 \end{pmatrix} = [\underline{w}]_{B'}$$

Check! $\underline{\underline{A}}\underline{v} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ -20 \end{pmatrix} = \underline{w}$

$[\underline{w}]_{B'} \rightarrow \underline{w}$

$$\begin{pmatrix} -27 \\ 36 \\ 7 \end{pmatrix} \rightarrow -27\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 36\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ -20 \end{pmatrix} \checkmark$$

19) a) $\underline{\underline{A}} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \; \underline{\underline{P}} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \; \underline{\underline{P}}^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$$\therefore \; \underbrace{\underline{\underline{A}}' = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}}_{\text{Similar}} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 5/3 & -1 \end{pmatrix}$$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$\therefore \underbrace{A' = P^{-1}AP}_{\hookrightarrow \text{Similar}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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c) $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

$$\therefore \underbrace{A' = P^{-1}AP}_{\hookrightarrow \text{Similar}} = \begin{pmatrix} \frac{7}{3} & \frac{10}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{4}{3} & \frac{8}{3} \\ \frac{2}{3} & -\frac{4}{3} & -\frac{2}{3} \end{pmatrix}$$

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20) a) $P[v]_B = Q[v]_{B'}$, Define $[v]_B = R[v]_{B'}$

$A = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}$ $\qquad$ $[v]_{B'} = R^{-1}[v]_B$

$\Longrightarrow R = P^{-1}Q$, $P = \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix}$, $Q = \begin{pmatrix} -12 & -4 \\ 0 & 4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & -2 & -12 & -4 \\ 3 & -2 & 0 & 4 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & -12 & -4 \\ 0 & 4 & 36 & 16 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & 9 & 4 \end{array}\right)$$

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$$\underline{\underline{R}}^{-1} = \begin{pmatrix} 4 & -4 \\ -9 & 6 \end{pmatrix} \frac{-1}{12}$$

$$\underline{\underline{A}}' = \underline{\underline{R}}^{-1} \underline{\underline{A}} \, \underline{\underline{R}} = \frac{-1}{12} \begin{pmatrix} 4 & -4 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 9 & 4 \end{pmatrix}$$

$$= \frac{-1}{12} \begin{pmatrix} 4 & -4 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} 36 & 20 \\ 36 & 16 \end{pmatrix} = \frac{-1}{12} \begin{pmatrix} 0 & 16 \\ -108 & -84 \end{pmatrix}$$

$$\underline{\underline{A}}' [\underline{v}]_{B'} = \frac{1}{12} \begin{pmatrix} 0 & 16 \\ -108 & -84 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 32 \\ -60 \end{pmatrix} = [\underline{w}]_{B'}$$

Check: $[\underline{v}]_B = \underline{\underline{R}} [\underline{v}]_{B'} = \begin{pmatrix} 6 & 4 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\underline{\underline{A}} [\underline{v}]_B = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} = [\underline{w}]_B$$

$$[\underline{w}]_{B'} = \underline{\underline{R}}^{-1} [\underline{w}]_B = \frac{-1}{12} \begin{pmatrix} 4 & -4 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \frac{-1}{12} \begin{pmatrix} 32 \\ -60 \end{pmatrix} \checkmark$$

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b) $$\underline{\underline{A}} = \begin{pmatrix} 3/2 & -1 & -1/2 \\ -1/2 & 2 & 1/2 \\ 1/2 & 1 & 5/2 \end{pmatrix}$$

$$\underline{\underline{P}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \underline{\underline{Q}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right), \quad \underline{\underline{R}}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\underline{\underline{A}}' = \underline{\underline{R}}^{-1} \underline{\underline{A}} \underline{\underline{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\underline{\underline{A}}' [\underline{v}]_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = [\underline{w}]_{B'}$$

Check: $[\underline{v}]_B = \underline{\underline{R}} [\underline{v}]_{B'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\underline{\underline{A}} [\underline{v}]_B = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = [\underline{w}]_B$$

$$[\underline{w}]_{B'} = \underline{\underline{R}}^{-1} [\underline{w}]_B = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$