Practice Problems 7

1) Verify that λ_i is an eigenvalue of $\boldsymbol{A}$ and that $\boldsymbol{x_i}$ is a corresponding eigenvector.

a) $\boldsymbol{A} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$, $\lambda_1 = -1, \boldsymbol{x_1} = (1,1)$, $\lambda_2 = 2, \boldsymbol{x_2} = (5,2)$

b) $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$, $\lambda_1 = 2, \boldsymbol{x_1} = (1,0,0)$, $\lambda_2 = -1, \boldsymbol{x_2} = (1,-1,0)$, $\lambda_3 = 3, \boldsymbol{x_3} = (5,1,2)$

2) Deteremine whether $\boldsymbol{x}$ is an eigenvector for $\boldsymbol{A}$ or not.

$$A = \begin{pmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{pmatrix}$$

a) $\boldsymbol{x} = (2,-4,6)$

b) $\boldsymbol{x} = (2,0,6)$

c) $\boldsymbol{x} = (2,2,0)$

d) $\boldsymbol{x} = (-1,0,1)$

3) Find the eigenvalues and corresponding eigenvectors.

a) $\boldsymbol{A} = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{pmatrix}$

4) Find the eigenvalues and corresponding eigenvectors for the transformation matrix then sketch/describe the effect on the unit square in $\mathbb{R}^2$ define by the unit vectors $\boldsymbol{u_1} = (1,0)$ and $\boldsymbol{u_2} = (0,1)$, (k is a constant).

a) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

5) Verify that $\boldsymbol{A}$ is diagonalisable by computing $\boldsymbol{P^{-1}AP}$.

a) $\boldsymbol{A} = \begin{pmatrix} -11 & 36 \\ -3 & 10 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} -3 & -4 \\ -1 & -1 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & -3 \\ 0 & 4 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

6) Show that the following matrices are not diagonalisable.

a) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

7) Determine whether the following matrices are diagonalisable or not. If they are then find an invertible matrix, $\boldsymbol{P}$, such that $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is diagonal.

a) $\boldsymbol{A} = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

d) $\boldsymbol{A} = \begin{pmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{pmatrix}$

8) Find a basis, B, for the domain of the linear transformation, T, such that the matrix of T relative to B is diagonal.

a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 : T(x, y) = (x + y, x + y)$

b) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 : T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$

9) If $\boldsymbol{A}$ is a diagonalisable matrix then there exists an invertible matrix, $\boldsymbol{P}$, such that $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP}$. Prove for a positive integer, k, that,

a) $\boldsymbol{B}^k = \boldsymbol{P}^{-1}\boldsymbol{A}^k\boldsymbol{P}$

b) $\boldsymbol{A}^k = \boldsymbol{P}\boldsymbol{B}^k\boldsymbol{P}^{-1}$

10) Use the result from question 9 to calculate the indicated power of $\boldsymbol{A}$.

a) $\boldsymbol{A} = \begin{pmatrix} 10 & 18 \\ -6 & -11 \end{pmatrix}$, $\boldsymbol{A}^6$

b) $\boldsymbol{A} = \begin{pmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{pmatrix}$, $\boldsymbol{A}^8$

11) Are the following matrices orthogonal or not?

a) $\boldsymbol{A} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

12) Find an orthogonal matrix that orthogonally diagonalises the following symmetric matrices and verify that $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$ gives the proper diagonal form.

a) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}$

13) Which of the following matrices are in Jordan normal form and which are not.

a) $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\boldsymbol{A} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}$

e) $\boldsymbol{A} = \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}$

f) $\boldsymbol{A} = \begin{pmatrix} 12 & 0 & 0 \\ 1 & 12 & 1 \\ 0 & 0 & 12 \end{pmatrix}$

g) $\boldsymbol{A} = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -9 \end{pmatrix}$

h) $\boldsymbol{A} = \begin{pmatrix} 5 & 10 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -9 \end{pmatrix}$

i) $\boldsymbol{A} = \begin{pmatrix} 12 & 1 & 0 \\ 0 & 12 & 1 \\ 0 & 0 & 12 \end{pmatrix}$

j) $\boldsymbol{A} = \begin{pmatrix} 12 & 1 & 0 \\ 0 & 12 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

14) Find the Jordan normal form of the following matrices.

a) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$