# Solutions 7

1) a) $\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda = -1$

b) $\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\lambda = 2$

$\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\lambda = -1$

$\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ $\lambda = 3$

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2) $\underline{A} = \begin{pmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{pmatrix}$

a) $\underline{A}\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ -16 \\ 24 \end{pmatrix} = 4\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$ eigenvector with $\lambda = 4$

b) $\underline{A}\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -16 \\ 12 \end{pmatrix}$ not an eigenvector

c) $\underline{A}\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ eigenvector with $\lambda = -2$

d) $\underline{A}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ eigenvector with $\lambda = -2$

3) a) $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} 6-\lambda & -3 \\ -2 & 1-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda) - 6 = \lambda^2 - 7\lambda = 0$$

$$\lambda(\lambda - 7) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 7$$

$\lambda_1 = 0$: $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector,

eigenspace: $\{(t, 2t) : t \in \mathbb{R}\}$

or $\text{Span}\{(1,2)\}$

$\lambda_2 = 7$: $\begin{pmatrix} -1 & -3 \\ -2 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector

eigenspace: $\text{Span}\{(-3,1)\}$

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b) $\underline{\underline{A}}$ is triangular $\therefore$ $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$

$\lambda_1 = 2$: $(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}) = 0 = \begin{pmatrix} 2-2 & 0 & 1 \\ 0 & 3-2 & 4 \\ 0 & 0 & 1-2 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{Span gives eigenspace}} \text{ is an eigenvector}$$

$\lambda_2 = 3$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector

$\lambda_3 = 1$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector

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c) $$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -2 & 5-\lambda & -2 \\ -6 & 6 & -3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & -2 \\ 6 & -3-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ 6 & -3-\lambda \end{vmatrix} - 6\begin{vmatrix} 2 & -2 \\ 5-\lambda & -2 \end{vmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(-3-\lambda)+12\right] + 2\left[2(-3-\lambda)+12\right] - 6\left[-4+2(5-\lambda)\right]$$

$$= -\lambda^3 + 3\lambda^2 + 9\lambda - 27$$

Factor theorem: $\lambda = -3 \Rightarrow 27 + 27 - 27 - 27 = 0 \therefore (\lambda+3)$ is a factor

$$= (\lambda+3)(-\lambda^2+6\lambda-9) = -(\lambda+3)(\lambda-3)^2 = 0$$

$\therefore \lambda_1 = -3,\ \lambda_2 = 3 \leftarrow$ multiplicity, 2

$\lambda_1 = -3$: $\begin{pmatrix} 4 & 2 & -2 \\ -2 & 8 & -2 \\ -6 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ is an eigenvector

$\lambda_2 = 3$: $\begin{pmatrix} -2 & 2 & -2 \\ -2 & 2 & -2 \\ -6 & 6 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Longrightarrow \underline{x} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are 2 eigenvectors,

eigenspace = Span$\{(1,1,0), (-1,0,1)\}$

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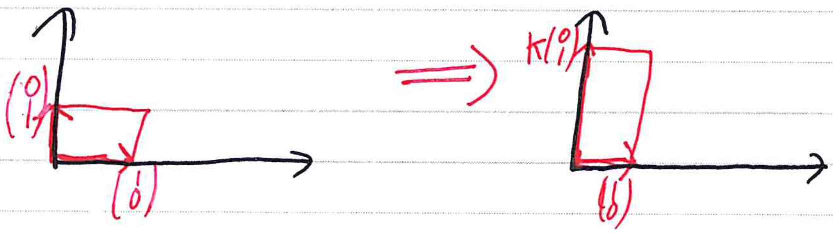
4) a) $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \Longrightarrow \lambda = 1, k$

$\lambda_1 = 1$: $\begin{pmatrix} 0 & 0 \\ 0 & k-1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector

$\lambda_2 = k$: $\begin{pmatrix} 1-k & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector




Vertical stretch

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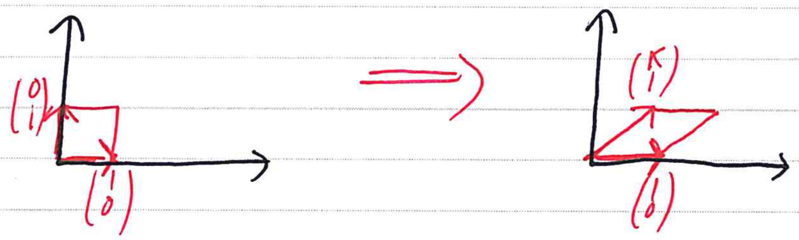
b) $\underline{\underline{A}} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \Longrightarrow \lambda = 1$

$\lambda = 1$: $\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ not an eigenvector $\Longrightarrow \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix}$




Horizontal shear

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c) $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \Longrightarrow \lambda = 1$

$\begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not an eigenvector: $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$


Vertical shear

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5) a) $|\underline{\underline{P}}| = -1, \quad \underline{\underline{P}}^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & 4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -1 & 3 \end{pmatrix}$

$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \begin{pmatrix} 1 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -11 & 36 \\ -3 & 10 \end{pmatrix}\begin{pmatrix} -3 & -4 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

b) $|\underline{\underline{P}}| = 12, \quad \underline{\underline{P}}^{-1} = \frac{1}{12}\begin{pmatrix} \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 4 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & -3 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} \end{pmatrix}$

$$\underline{P}^{-1} = \frac{1}{12}\begin{pmatrix} 8 & -8 & 12 \\ 0 & 3 & 0 \\ -4 & 1 & 0 \end{pmatrix}$$

$$\underline{P}^{-1}\underline{A}\,\underline{P} = \frac{1}{12}\begin{pmatrix} 8 & -8 & 12 \\ 0 & 3 & 0 \\ -4 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 & -3 \\ 0 & 4 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 8 & -8 & 12 \\ 0 & 3 & 0 \\ -4 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 3 & 3 \\ 0 & 12 & 0 \\ 5 & 6 & -2 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 60 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & -12 \end{pmatrix}$$

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6) a) $\underline{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies \lambda = 1$ multiplicity, 2

$2\times 2$ matrix $\implies n = 2$

$(\underline{A} - \lambda \underline{I}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector,

Only 1 linearly independent eigenvector $< n$

$\therefore$ not diagonalisable.

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b) $\underline{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \implies \lambda = 1, 2$ multiplicity, 2

$3\times 3$ matrix $\implies n = 3$

$\lambda = 1$: $(\underline{A} - \lambda_1 \underline{I}) = \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\implies \underline{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector

$\lambda = 2$: $\begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector

Only 2 linearly independent eigenvectors $< n$

$\therefore$ not diagonalisable.

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7) a) From Q 3a we know 2 linearly independent eigenvectors, $(1,2)$, $(-3,1)$. we make $\underline{P}$ using the eigenvectors as columns.

$$\underline{P} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}, \quad ||\underline{P}|| = 7, \quad \underline{P}^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$$

$$\underline{P}^{-1}\underline{\underline{A}}\,\underline{P} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 0 & 0 \\ 0 & 49 \end{pmatrix}$$

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b) $$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right]$$

$$= -\lambda^3 + 7x^2 - 14x + 8$$

$$= -(\lambda-1)(\lambda-2)(\lambda-4)$$

$$\therefore \lambda = 1, 2, 4$$

$\lambda = 1$: $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

eigenvector

$\lambda=2$: $\begin{pmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

eigenvector

$\lambda=4$: $\begin{pmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$

$\therefore \underline{P} = \begin{pmatrix} -1 & 1 & 7 \\ 1 & 0 & -4 \\ 1 & 0 & 2 \end{pmatrix}, \quad \|\underline{P}\| = -6, \quad \underline{P}^{-1} = \frac{-1}{6}\begin{pmatrix} 0 & -2 & -4 \\ -6 & -9 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$$\underline{P}^{-1}\underline{A}\,\underline{P} = \frac{-1}{6}\begin{pmatrix} 0 & -2 & -4 \\ -6 & -9 & 3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 7 \\ 1 & 0 & -4 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \frac{-1}{6}\begin{pmatrix} -6 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -24 \end{pmatrix}$$

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c) $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(2-\lambda)\right] \Rightarrow \lambda = 1, 2$ multiplicity 2

$\lambda=1$: $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda=2$: $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Only 2 linearly independent eigenvectors for $3\times3$ matrix

$\therefore$ not diagonalisable.

d) $\begin{vmatrix} -\lambda & -3 & 5 \\ -4 & 4-\lambda & -10 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\left[-\lambda(4-\lambda)-12\right]$

$= -\lambda^3 + 8\lambda^2 - 4\lambda - 48$

$= -(\lambda+2)(\lambda-4)(\lambda-6)$

$\therefore \lambda = -2, 4, 6$

$\underline{\lambda=-2}$: $\begin{pmatrix} 2 & -3 & 5 \\ -4 & 6 & -10 \\ 0 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

$\underline{\lambda=4}$: $\begin{pmatrix} -4 & -3 & 5 \\ -4 & 0 & -10 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -5 \\ 10 \\ 2 \end{pmatrix}$

$\underline{\lambda=6}$: $\begin{pmatrix} -6 & -3 & 5 \\ -4 & -2 & 10 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$\underline{\underline{P}} = \begin{pmatrix} 3 & -5 & -1 \\ 2 & 10 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad |\underline{\underline{P}}| = -16, \quad \underline{\underline{P}}^{-1} = \frac{-1}{16}\begin{pmatrix} -4 & -2 & 0 \\ 0 & 0 & -8 \\ 4 & -6 & 40 \end{pmatrix}$

$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \frac{-1}{16}\begin{pmatrix} -4 & -2 & 0 \\ 0 & 0 & -8 \\ 4 & -6 & 40 \end{pmatrix}\begin{pmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & -5 & -1 \\ 2 & 10 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

$= \frac{-1}{16}\begin{pmatrix} 32 & 0 & 0 \\ 0 & -64 & 0 \\ 0 & 0 & -96 \end{pmatrix}$

8) a) Standard matrix, $\underline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2)$$

$$\Rightarrow \lambda = 0, 2$$

$\lambda=0$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda=2$: $\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $|\underline{P}| = -2$, $\underline{P}^{-1} = \frac{-1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$

$\underline{D} = \underline{P}^{-1}\underline{A}\,\underline{P} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$

is then matrix of $T$ relative to $B$ where,

$B = \{(-1,1), (1,1)\}$

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b) Standard matrix, $\underline{A} = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = (-2-\lambda)\left[-(1-\lambda)\lambda - 12\right]$$

$$-2\left[-2\lambda - 6\right] + 3\left[-4 + (1-\lambda)\right]$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$= -(\lambda+3)^2(\lambda-5) \Rightarrow \lambda = -3, 5$$

multiplicity 2

$\lambda=-3$: $\begin{pmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \underline{x} = x_2\begin{pmatrix}-2\\ 1\\ 0\end{pmatrix} + x_3\begin{pmatrix}3\\ 0\\ 1\end{pmatrix}$

$\lambda=5$: $\begin{pmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix}-1\\ -2\\ 1\end{pmatrix}$

$$\underline{\underline{P}} = \begin{pmatrix}-2 & 3 & -1\\ 1 & 0 & -2\\ 0 & 1 & 1\end{pmatrix}, \quad |\underline{\underline{P}}| = -8, \quad \underline{\underline{P}}^{-1} = -\frac{1}{8}\begin{pmatrix}2 & -4 & -6\\ -1 & -2 & -5\\ 1 & 2 & -3\end{pmatrix}$$

$$\underline{\underline{D}} = -\frac{1}{8}\begin{pmatrix}2 & -4 & -6\\ -1 & -2 & -5\\ 1 & 2 & -3\end{pmatrix}\begin{pmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix}\begin{pmatrix}-2 & 3 & -1\\ 1 & 0 & -2\\ 0 & 1 & 1\end{pmatrix}$$

$$= -\frac{1}{8}\begin{pmatrix}24 & 0 & 0\\ 0 & 24 & 0\\ 0 & 0 & 40\end{pmatrix}$$

is matrix of $T$ relative to $B$

where, $B = \{(-2,1,0),(3,0,1),(-1,-2,1)\}$

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9) a) $$\underline{\underline{B}}^K = (\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})^K = \underbrace{(\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})(\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})\cdots(\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})}_{K \text{ times}}, \quad \underline{\underline{P}}\,\underline{\underline{P}}^{-1} = \underline{\underline{I}}$$

Group each $\underline{\underline{A}}\,\underline{\underline{P}}\,\underline{\underline{P}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}^2$ etc,

$$= \underline{\underline{P}}^{-1}\underline{\underline{A}}^K\underline{\underline{P}}$$

b) $\underline{\underline{B}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} \Rightarrow \underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{B}}\,\underline{\underline{P}}^{-1}$

Using the same argument as part a) it is easy to see that $\underline{\underline{A}}^K = \underline{\underline{P}}\,\underline{\underline{B}}^K\underline{\underline{P}}^{-1}$.

10) a) Using the same technique the eigenvectors are,

$(-3, 2)$ and $(-2, 1)$

$\Rightarrow \underline{P} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}, \quad \underline{P}^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$

$\underline{B} = \underline{P}^{-1} \underline{A} \underline{P} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

$\underline{B}^6 = \begin{pmatrix} (-2)^6 & 0 \\ 0 & 1^6 \end{pmatrix} = \begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \underline{A}^6 = \underline{P} \underline{B}^6 \underline{P}^{-1} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -188 & -378 \\ 126 & 253 \end{pmatrix}$

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b) Eigenvectors of $\underline{A}$ are; $(-1, 3, 1), (3, 0, 1), (-2, 1, 0)$

$\underline{P} = \begin{pmatrix} -1 & 3 & -2 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \underline{P}^{-1} = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{5}{2} \\ -\frac{3}{2} & -2 & \frac{9}{2} \end{pmatrix}$

$\underline{B} = \underline{P}^{-1} \underline{A} \underline{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \underline{B}^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 256 \end{pmatrix}$

$\therefore \underline{A}^8 = \underline{P} \underline{B}^8 \underline{P}^{-1} = \begin{pmatrix} 384 & 256 & -384 \\ -384 & -512 & 1152 \\ -128 & -256 & 640 \end{pmatrix}$

11) a) $\underline{v}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}^T$, $\underline{v}_2 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}^T$

$\underline{v}_1 \cdot \underline{v}_2 = 0$, $\|\underline{v}_1\| = \|\underline{v}_2\| = 1$

Columns form an orthonormal set. $\therefore$ $\underline{A}$ is orthogonal.

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b) $\underline{v}_1 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}^T$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T$, $\underline{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}^T$

$\underline{v}_1 \cdot \underline{v}_2 = \underline{v}_1 \cdot \underline{v}_3 = \underline{v}_2 \cdot \underline{v}_3 = 0$

but $\|\underline{v}_1\| = \|\underline{v}_3\| = 5 \neq 1$ $\therefore$ not orthonormal.

$\therefore$ $\underline{A}$ is not orthogonal.

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12) a) Eigenvectors are $(1,-1)$ and $(1,1)$.

The eigenvectors are orthogonal already so just need to normalise: $\underline{c}_1 = \frac{1}{\sqrt{2}}(1,-1)$, $\underline{c}_2 = \frac{1}{\sqrt{2}}(1,1)$

$\therefore$ $\underline{P} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ is orthogonal. $\therefore$ $\underline{P}^{-1} = \underline{P}^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

$$\underline{P}^T \underline{A} \underline{P} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

(Diagonal)

b) Eigenvectors of $\underline{\underline{A}}$ are $(-2,1,2)$, $(-1,2,-2)$, $(2,2,1)$.

They are orthogonal so just need to normalise:

$$\underline{e}_1 = \frac{1}{3}(-2,1,2),\ \underline{e}_2 = \frac{1}{3}(-1,2,-2),\ \underline{e}_3 = \frac{1}{3}(2,2,1)$$

$$\underline{\underline{P}} = \frac{1}{3}\begin{pmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$ is orthogonal $\therefore \underline{\underline{P}}^{-1} = \underline{\underline{P}}^T$

$$\underline{\underline{P}}^T = \frac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\underline{\underline{P}}^T \underline{\underline{A}}\,\underline{\underline{P}} = \begin{pmatrix} -15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

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13) a) Jordan

b) Not Jordan

c) Jordan

d) Jordan

e) Not Jordan

f) Not Jordan

g) Jordan

h) Not Jordan

i) Jordan

j) Not Jordan

14) a) Already Jordon normal form,

b) Rank 1: $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$$

$\lambda = 1$ multiplicity, 3

$$\underline{\underline{A}} - \underline{\underline{I}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \underline{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Rank 2: $(\underline{\underline{A}} - \underline{\underline{I}})^2 = \underline{\underline{0}}$ (zero matrix)

$$\Longrightarrow \underline{x} = \underbrace{x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{This one is linearly dependent on a rank 1 eigenvector}} + \underbrace{x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{Choose this as seed,}} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let, $\underline{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = (\underline{\underline{A}} - \underline{\underline{I}}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Which leaves $\underline{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Longrightarrow \underline{\underline{T}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$\underline{\underline{T}}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \therefore \underline{\underline{J}} = \underline{\underline{T}}^{-1} \underline{\underline{A}} \underline{\underline{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

c) Rank 1: $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{vmatrix} = -\lambda^3 + 11\lambda^2 - 39\lambda + 45$$
$$= -(\lambda-3)^2(\lambda-5)$$

$$\lambda = 3, 5$$

multiplicity, 2

$$(\underline{\underline{A}} - 3\underline{\underline{I}}) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

2 linearly independent eigenvectors already and we still have another eigenvalue which will give us another eigenvector

$\therefore$ don't need any more generalised eigenvectors.

$$(\underline{\underline{A}} - 5\underline{\underline{I}}) = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\underline{\underline{T}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad \underline{\underline{T}}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\underline{\underline{J}} = \underline{\underline{T}}^{-1} \underline{\underline{A}}\, \underline{\underline{T}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$