Linear Algebra Survival Guide

An essentials only approach

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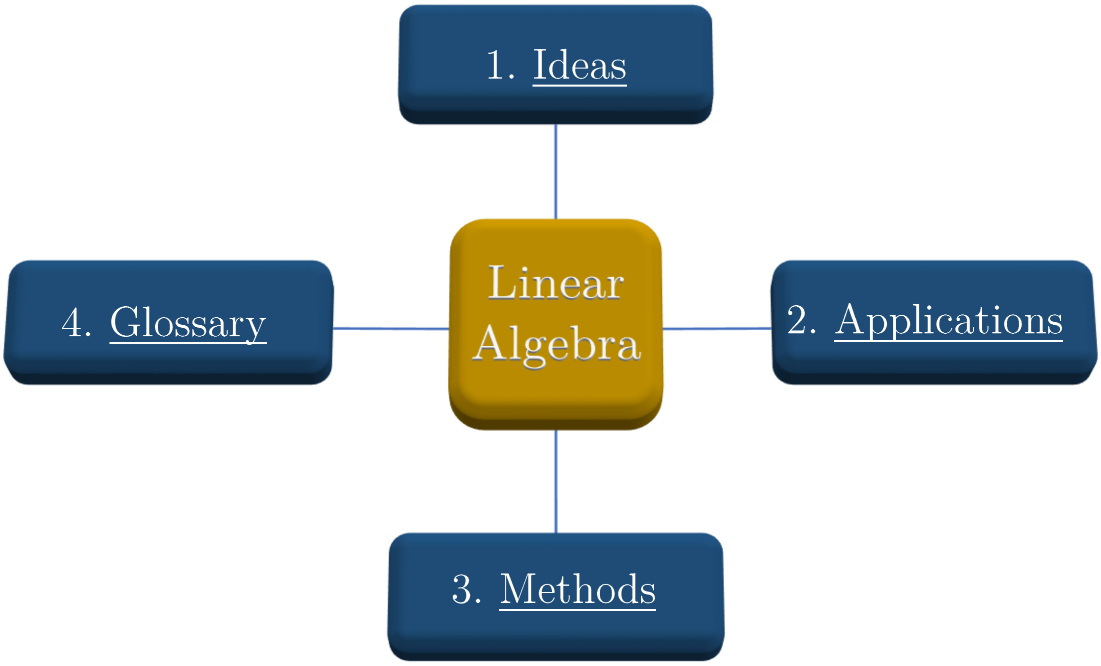
What this guide contains...

- The main ideas of linear algebra as a whole and by individual lecture.
- Direct applications of linear algebra so you can see how it is used.
- An "essentials only" approach to using the methods and understanding the terms.

Click on the hyperlinks (underlined) embedded within the text to navigate.



1. Ideas

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This section goes through the main ideas behind linear algebra and those contained in the lectures.

- General linear algebra
- Lecture 1
- Lecture 2
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- Lecture 7

- The main idea is to develop methods to manipulate and solve systems of linear equations.

- You will see that many applications involve linear systems, including certain analysis of nonlinear systems.

- Since real-world applications often have extremely large systems we generally utilise computer code to solve them. This computer code must be made as efficient as possible, particularly in the age of big data. The theories of linear algebra are what allow us to make the code efficient for the specific application.

- There is a lot of manipulation of matrices because matrices are a convenient way of storing data (think about making tables to store information by grouping things in various columns and rows – a matrix is just like a table). We therefore develop techniques to manipulate matrices in order to get the information we need out of the data.

- In the lower dimensions the vector algebra we do corresponds directly with geometry.

- Non-physics based problems in higher dimensions generally have no geometric equivalent. We must therefore be able to understand and manipulate the higher dimensional functions/systems without the need for picturing the geometry – in other words to rely on the mathematics rather than pictures. However, pictures in the lower dimensions sometimes help us understand what is going on in the higher dimensions through analogy.

- Finally, vectors are generally quite easy to manipulate compared with general functions and operators you tend to find in statistics/calculus/differential equations. Imagine if we could map more complicated problems into some kind of space to do with vectors… Say, a vector space for example… Then we could manipulate/solve more easily whilst preserving the system.

Linear systems

- Equations with only constants and x^1 terms.
- No functions like $\sin(x)$ or x^2 etc.
- Can represent linear systems in matrix form: $\boldsymbol{Ax} = \boldsymbol{b}$.
- Form augmented matrix: $\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$
- Pretend straight line is column vector $\boldsymbol{x}$ like: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Echelon Form

- Every matrix has many echelon forms.
- The form is useful for extracting solutions from the matrix.
- Recognise using the following method:
  - Start with top row at the left → Find first non-zero value → This is your first pivot → Everything below the pivot should be 0.
  - Move down to second row → Find first non-zero row → This is your next pivot → Everything below should be 0.
  - Repeat for all rows.

Reduced Echelon Form (REF)

- Same as echelon form but each pivot = 1 and everything below AND above should be 0.
- Every matrix has a unique REF.

Gaussian Elimination

- The process of obtaining an echelon form.

Gauss-Jordan Elimination

- The process of obtaining the REF.

Basic/Free Variables

- Basic variables correspond with the pivot positions.
- Free variables can be any value.

  $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ → x_1, x_3 basic, x_2 free

  Pivot columns 1 & 3

Homogeneous Systems

$$\boldsymbol{Ax} = \boldsymbol{0}$$

Non-homogeneous Systems

$$\boldsymbol{Ax} = \boldsymbol{b}, \qquad \boldsymbol{b} \neq \boldsymbol{0}$$

- Solution, $\boldsymbol{x}$, is sum of a particular solution and the solution to the homogeneous part.

Elementary Matrices

- Perform 1 row operation on the identity matrix to get an elementary matrix.
- Doing a row operation to a matrix is the same as left-multiplying it by the elementary matrix: $\boldsymbol{EA}$
- Can write an invertible matrix as the product of elementary matrices.

Inverse of a Matrix

- Can calculate using Gauss-Jordan elimination or using the adjugate formula.

LU-Factorisation

- $\boldsymbol{L}$ means lower triangular matrix: $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- $\boldsymbol{U}$ means upper triangular matrix: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$

- Can factor a matrix into a product $\boldsymbol{LU}$.
- Can use it to solve linear systems more efficiently than Gauss-Jordan elimination.

Vector Spaces

- A collection of objects that obey 10 rules.
- The objects may be vectors (hence the name) or the objects can be anything else that obeys the 10 rules.

  E.g. The set of continuous functions on an interval, $C[a, b]$.

  The set of polynomials of degree n or less, P_n.

  The set of $m \times n$ general matrices, M_{mn}.
- Any methods we apply to vector spaces can equally apply to general vector spaces such as continuous functions etc.

Subspaces

- Subset of a vector space that is closed under addition and scalar multiplication. In other words adding two objects together results in an object from the same space. Likewise with scalar multiplication.

Linear Combination

- Adding multiples of vectors together: $3\boldsymbol{v_1} - 5\boldsymbol{v_2} + \cdots$

Linear Independence

- When vectors are not linear combinations of each other.
- Can test using Gauss-Jordan elimination.
- If the number of vectors is greater than the dimension of the vector the set will definitely be linearly dependent.

Span

- If we have a set of vectors from a vector space, V, we call it a spanning set (or just span) if ***every vector*** in V can be written as a linear combination of the spanning set vectors.
- The number of linearly independent vectors in a spanning set dictate the dimension of real space it spans

  E.g. $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ spans $\mathbb{R}^3$ if the vectors are linearly independent.

  $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ spans $\mathbb{R}^2$ if the vectors are linearly independent.
- All spanning sets are subspaces.
- Every vector in V can be written as a unique linear combination of the spanning vectors.

Basis

- A linearly independent spanning set.

Standard Basis

- $\mathbb{R}^2$: $\{(1,0),(0,1)\}$
- $\mathbb{R}^3$: $\{(1,0,0),(0,1,0),(0,01)\}$
- M_{22}: $\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$

  Etc.

Row Space

- Span of the row vectors in a matrix.

Column Space

- Span of the column vectors in a matrix.

Null Space

- Span of the homogeneous equation solutions, $\boldsymbol{Ax} = \mathbf{0}$.

Dimension of a Vector Space

- The number of basis vectors.

Finding a Basis

- Start with a spanning set then make linearly independent by removing vectors.

Basis for Null Space

- Span of solution to homogeneous equation.

Basis for Row Space

- Non-zero rows of echelon form.

Basis for Column Space

- Non-zero rows of echelon form of transpose.

Or,

- Find linearly independent columns from echelon form → Use corresponding original columns as basis.

Rank

- Dimension of the row space (same as dimension of the column space).

Nullity

- Dimension of the null space.

Rank-Nullity Relationship

- Rank + nullity = number of columns in matrix

Meaning of Column Space

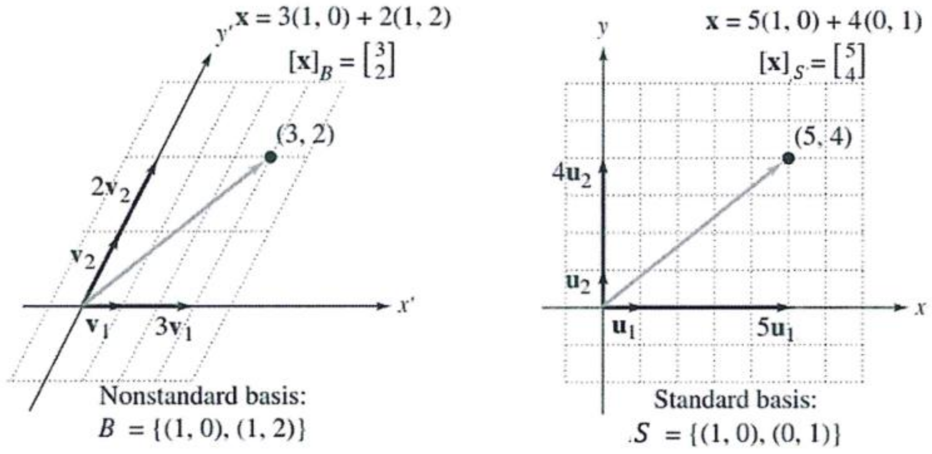
- The column space gives you every vector, $\boldsymbol{b}$, that will have a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Coordinates

- Write a vector as a linear combination of basis vectors → The coefficients (constants) are called the coordinates relative to that basis.

Change of Basis

- In lower dimensions it is like changing coordinate systems:



- Can change basis by solving a linear system or by using a transition matrix.
- Coordinates in the standard basis can be written as $[\boldsymbol{v}]_S$ or simply $\boldsymbol{v}$.
- Coordinates in a non-standard basis, B, can be written as $[\boldsymbol{v}]_B$.

Transition Matrices

- From a basis, B, to the standard basis it is just made up from the basis vectors in B. Use them as columns.
- Can convert between 2 non-standard bases using Gauss-Jordan elimination.

Vectors in Higher Dimensions

- All the usual formulas still hold but just extend the ideas to include all elements.

Cauchy-Schwarz Inequality

- Used to prove we can still define the angle between higher dimension vectors
- Helps prove the triangle inequality.

Triangle Inequality

- Generalised Pythagoras' theorem.
- Used in many areas of mathematics.

Inner Product

- An operation that takes 2 inputs and obeys 4 rules.
- Generalises the dot product and are used to derive Fourier series in calculus, and speeding up computations using Krylov subspaces (not covered in this course).
- The regular dot product is known as the Euclidean inner product. Use this one for vector spaces $\mathbb{R}^n$ if the inner product is not specified.

Orthogonality

- Vectors are orthogonal if their inner product is 0.

Orthogonal Projection onto Vectors

- In lower dimensions it is like the "shadow" of a vector on another vector.
- Represents the closest distance between 2 vectors.
- This fact is used to do least squares.

Orthogonal Sets

- All vector combinations in both sets are orthogonal to each other.

Orthonormal Vectors

- Orthogonal vectors which are also unit vectors.
- Orthogonal/orthonormal vectors are also linearly independent.

Orthonormal Basis

- Basis consisting of orthonormal vectors.
- Gives a very convenient way to write coordinates of any vector (use inner product).
- The $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ vectors on the x, y, z axes in the Cartesian coordinate system are an orthonormal basis.
- The coordinates are then just the projections of the vector onto the basis vectors.
- Since the norms are 1 these projections are just the inner products (projection is inner product divided by norm).

Orthogonal Matrix

- A matrix with row vectors that are orthonormal.
- The columns will also be orthonormal.

Gram-Schmidt Orthonormalisation

- Method of constructing an orthonormal basis from a general basis.
- Start with the first basis vector then build from there by subtracting consecutive projections.

Orthogonal Subspaces

- Every combination of vectors from 2 subspaces is orthogonal.

Orthogonal Complement

- Set of all vectors which are orthogonal to every vector in a vector space.

Projection onto Subspace

- Generalisation of vector projection.
- Used for least squares.

Least Squares

- Used to get best approximate solution to an inconsistent system.
- Inconsistent systems occur all the time when you take measurements of anything due to errors.
- Minimises the total error.

Linear Transformation

- A function from one vector space to another that can be split with regard to addition and have a scalar multiple factored out.
- Note that not all straight lines are linear transformations even though we call them linear graphs – only the straight lines through the origin are linear transformations.

Image of a Vector

- Just apply the transformation to the vector.

Pre-image of a vector

- Find all vectors which get transformed into the vector.

Domain

- Set of inputs.

Range

- Set of outputs for each input.

Codomain

- The vector space which contains the outputs.
- It is all possible outputs and is either greater than or equal to the range.

Transforming Basis Vectors

- If we know how a transformation affects the basis vectors, we can find the transformation of any vector in the span of that basis.

Matrices for Linear Transformations

- All matrices can be considered as a linear transformation.
- We like these because matrices are convenient for data storage and easy to manipulate.
- The column space gives the range in this case.

Kernel

- All vector which get mapped to the 0 vector.
- If the kernel is only the zero vector then the transformation is one-to-one.

Rank/Nullity

- Rank is the dimension of the range.
- When the rank equals the dimension of the codomain the transformation is onto (range = codomain).
- Nullity is the dimension of the kernel.

- Nullity is 0 for one-to-one transformations (dimension of the set with only the zero vector is 0).
- Rank + nullity = dimension of codomain (similar to simple matrix systems)

Isomorphisms (see application)

- Vector spaces which are basically the same but just written differently.
- If the dimensions of the vector spaces are equal then they are isomorphic.

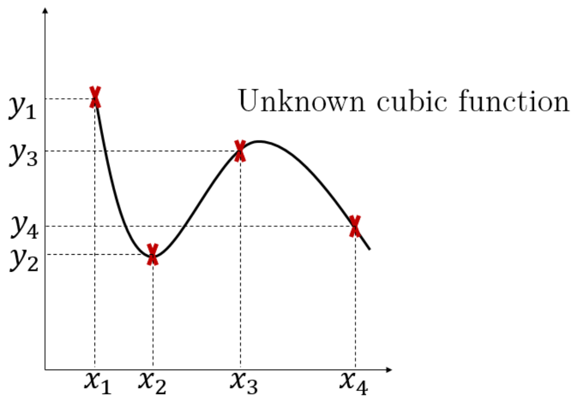
E.g.

- $P_3 = a_0 + a_1x + a_2x^2 + a_3x^3$ has standard basis $\{1, x, x^2, x^3\}$.
- Even though the vector space represents a system of nonlinear equations, the coefficients with respect to the basis are just vectors in $\mathbb{R}^4$:

$$(a_0, a_1, a_2, a_3) = (1,0,-1,2)$$

→ The vector $(1,0,-1,2)$ fully describes the cubic polynomial, $1 - x^2 + 2x^3$.

→ Now imagine you have 4 data points and you want to fit a cubic curve to it:



$$P_3 = a_0 + a_1x + a_2x^2 + a_3x^3$$

We know the values of each (x_i, y_i) leading to,

$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3$$

$$y_2 = a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3$$

$$y_3 = a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^3$$

$$y_4 = a_0 + a_1x_4 + a_2x_4^2 + a_3x_4^3$$

So we just solve,

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

To get the a_i (see applications section).

We can utilise linear algebra here because P_3 is isomorphic to $\mathbb{R}^4$.

Standard Matrix

- For a transformation relative to the standard basis it is just the variable coefficients.
- If the transformation is relative to a general basis then we transform the basis vectors and use them as columns to get the standard matrix.

Composite Transformations

- Make sure dimensions match.
- Can represent as multiplying matrices together from right to left (the first transformation has matrix closest to the vector).

Inverse Transformations

- Just find inverse matrix then use them as coefficients of the variables.

Matrix of $\boldsymbol{T}$ Relative to $\boldsymbol{B}$ and $\boldsymbol{B'}$

- Imagine we have a transformation from the standard basis to the standard basis:
$$T(\boldsymbol{v}) = \boldsymbol{w}$$
- In many real-world problems we are given measurements with respect to a different coordinate system than our own. Sometimes we are also required to give the result in yet another coordinate system.
How can we represent the original transformation but between the 2 different coordinate systems?
$$T([\boldsymbol{v}]_B) = [\boldsymbol{w}]_{B'}$$
We need a matrix, $\boldsymbol{A}^*[\boldsymbol{v}]_B = [\boldsymbol{w}]_{B'}$
This is the matrix of $\boldsymbol{T}$ relative to B and B'.
- We have 2 techniques – one for square matrices and one for non-square.

Similar Matrices

- We might know a transformation for a basis, B.
- We then might have some data in a different basis, B'.
- We want to apply the transformation to our data but the problem is it is with respect to B.
- Normally we would have to change from B' to B → Transform → Convert back to B'.
- We can find a similar matrix, $\boldsymbol{A'}$, relative to B' instead to avoid converting coordinates all the time.

Eigenvalues/Eigenvectors

- Vectors, $\boldsymbol{x}$, and numbers, λ, that satisfy $\boldsymbol{Ax} = \lambda\boldsymbol{x}$.
- Represents a stretch or contraction but the direction of the vector is unchanged by $\boldsymbol{A}$.
- All other vectors that are not eigenvectors will have their direction changed.
- Each eigenvector has a corresponding eigenvalue.
- We find them by solving the characteristic equation, $|\boldsymbol{A} - \lambda\boldsymbol{x}| = 0$.
- Repeated roots of the equation correspond with the multiplicity of an eigenvalue.

Eigenspace

- Union of all eigenvectors for a particular eigenvalue with the zero vector.

Diagonalisation

- We can make a matrix diagonal by finding a special matrix.
- Not all matrices are diagonalisable.
- Diagonal matrices are easy to work with which is why we want them (check out the application for finding powers of a matrix).
- An $n \times n$ matrix is diagonalisable if it has n linearly independent eigenvectors.
- If we have a linear transformation represented by a matrix, we can find a basis so that the matrix is diagonal which simplifies future calculations.
- If we can diagonalise a system of linear differential equations then the solutions become easy to handle.

Orthogonal Diagonalisation

- The same as diagonalisation but in an orthonormal basis.
- This can be done on symmetric matrices only.
- We get the same benefits of diagonalisation with the added bonus that all coordinates are also easy since the basis is orthonormal.

Jordan Normal Form

- A block matrix which has non-zero leading diagonal blocks and 1's on the diagonal just above the leading diagonal of a block.
- If a block is 1×1 then there are no 1's in that block.
- All matrices have a unique Jordan normal form.
- The matrix used to obtain it is not necessarily unique.
- The uniqueness of the Jordan normal form is used in further linear algebra topics which lead to methods of solving other problems.

Generalised Eigenvectors

- Eigenvectors of the characteristic equation raised to powers.
- Used to find the Jordan normal form.

2. Applications

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This section contains a small selection of applications which utilise topics from across the lectures. Many more exist, particularly with regard to probability and statistics, however as I am not a statistician I will save those for your statistics lecturers.

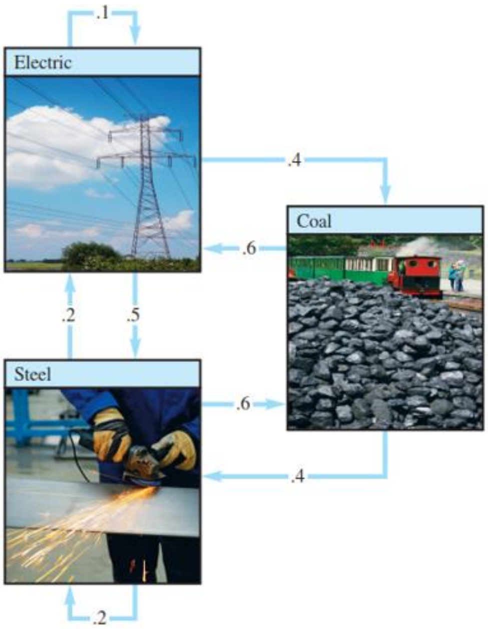
The necessary background for some of these applications is left out since the purpose is just to let you see how these things are applied. If you are interested in where the equations come from and the details of what they mean then you can self-study whatever you are interested in.

I also have resources for each of the topics which can be made available on request.

- Solving large systems of linear equations.
- Fitting polynomial curves to data points.
- Finding lines of best fit for inconsistent data.
- Solving systems of linear ordinary differential equations.
- Using eigenvalues to understand complicated nonlinear systems of ordinary differential equations.
- Writing efficient computer code to solve large systems.
- Probability.

- In 1949 Professor Wassily Leontief was working an economic model for the US economy.
- He used 250,000 pieces of information from the Bureau of Labour and Statistics to formulate a model of 500 economic sectors, including coal, automotive, communications etc.
- Large amounts of data like this are usually modelled as linear systems since the computation task is very difficult.
- Leontief's model had 500 equations that linked each sectors output to other sectors of the economy resulting in a system of 500 linear equations in 500 unknowns.
- Repeatedly solving such systems when changing parameters (such as production rates etc.) consumes a lot of time and so we seek efficient algorithms.
- *LU*-factorisation is more efficient than Gaussian/Gauss-Jordan elimination and the marginal gains at the small scale are amplified at the large scale.

Simplified model:



- Read down columns to see where output goes.
- Read across rows to see required inputs

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

- Equilibrium prices are when each sectors incomes equal their expenditures.
- The linear system to find these prices is:

$$\begin{aligned} p_C - .4p_E - .6p_S &= 0 \\ -.6p_C + .9p_E - .2p_S &= 0 \\ -.4p_C - .5p_E + .8p_S &= 0 \end{aligned}$$

- We write in matrix form, row reduce and get the equilibrium price vector:

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

- An earlier section discussed how to do this for a cubic. We can map the nonlinear cubic problem, $a_0 + a_1x + a_2x^2 + a_3x^3$, to the isomorphic vector space $(a_0, a_1, a_2, a_3) \in \mathbb{R}^4$:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

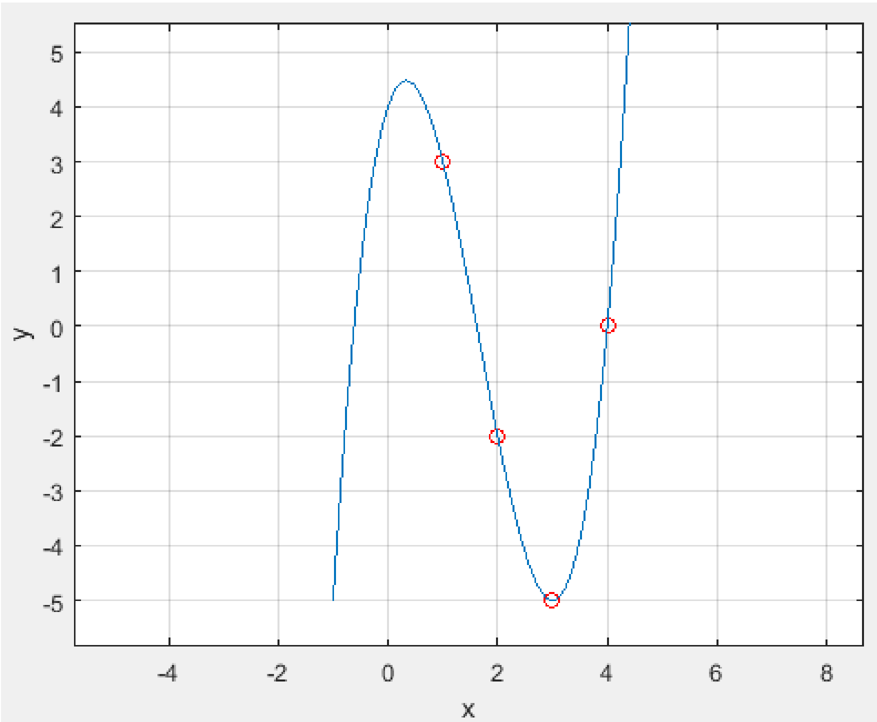
  For known points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4).

E.g. Find the cubic polynomial that passes through (1,3), (2,-1), (3,-5) and (4,0).

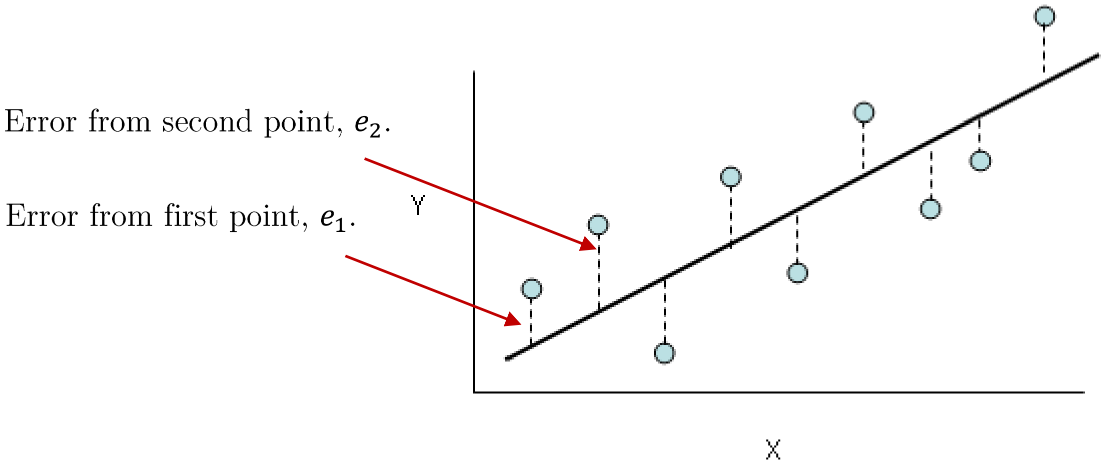
i	x_i	y_i
1	1	3
2	2	-2
3	3	-5
4	4	0

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \rightarrow (a_0, a_1, a_2, a_3) = (4,3,-5,1)$$

So the cubic that fits the points is $p(x) = 4 + 3x - 5x^2 + x^3$.



- In lecture 5 we derived the least squares formula for finding lines of best fit (regression lines):
$$\boldsymbol{A^T A x = A^T b}$$
- This equation came from trying to minimise the norm of the error between the calculated line and the data points.
- The norm of the error represents the total error of our line approximation across all points.



We are trying to minimise the total error which is given by,

$$\sqrt{e_1^2 + e_2^2 + \cdots}$$

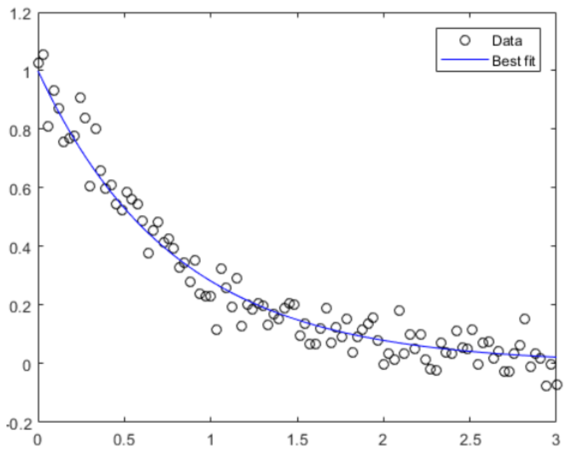
We square each error so the positive and negative errors don't cancel out. Then add them up and square root them to get a representation of the total error. This is the definition of norm for an "error vector" $= (e_1, e_2, \ldots)$.

If we minimise this then we have found the best possible fit.

We construct the error vector to be an orthogonal projection which we know minimises the distance resulting in $\boldsymbol{A^T A x = A^T b}$.

We solve the resulting matrix equation using standard linear algebra techniques.

It is also possible to formulate linear approximations to do nonlinear least squares fits that can still be solved using Gaussian elimination etc. such as the graph below.



- We already know how to solve a linear 1st order differential equation, but what happens when we have large systems of them? This occurs frequently across all industries.
- In a similar way to how we mapped the vector space of polynomials to a simple vector space in $\mathbb{R}^n$ (isomorphisms), we can do the same for systems of linear differential equations.

E.g.

$$\frac{dx_1}{dt} = 4x_1 + 7x_2$$

$$\frac{dx_2}{dt} = -2x_1 - 5x_2$$

This can be written as,

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \boldsymbol{x}' = \boldsymbol{Ax}$$

It can be shown that to solve this system we require,

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$$

Which is the eigenvalue problem.

Once we get the eigenvalues and eigenvectors the solutions will be of the form,

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t}$$

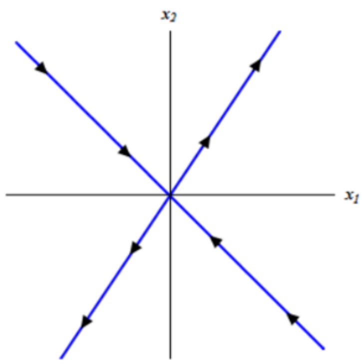
In our example the eigenvalues/eigenvectors are:

$$\lambda_1 = -1, \quad \boldsymbol{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -4, \quad \boldsymbol{v_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

We can see from the eigenvectors that the solutions are linearly independent. Therefore we can make a new solution by adding them together as a linear combination:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}$$

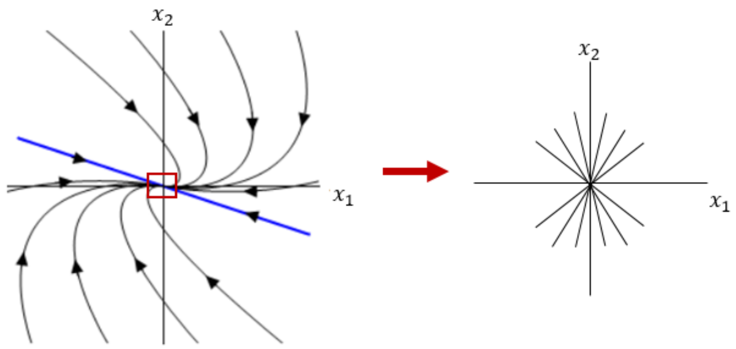
Choosing different values of each c_i give different curves in the (x_1, x_2)-plane. Selecting the curves when $c_1 = 0$, then when $c_2 = 0$, we get 2 straight lines. We can tell which direction they move in because of their eigenvalues (positive or negative exponent):



- Nonlinear systems of ODEs are frequent occurrences in real-life applications:

$$\frac{dx_1}{dt} = \cdots$$
$$\frac{dx_2}{dt} = \cdots$$
$$\frac{dx_3}{dt} = \cdots$$
$$\vdots$$

- We can even analyse these using linear algebra by considering the equilibrium points (when rates of change are 0) and making linear approximations around them:



- We take the Jacobian around an equilibrium point as a linearisation of the system and solve the eigenvalue equation again to get the trajectories of the solution (away or towards the point).

- We can also diagonalise the matrix in some cases which represents a decoupling of the system variables (Can re-write system as 2 independent equations with new variables).

- This diagonalisation is the equivalent of changing coordinates to a basis consisting of the eigenvectors.

- When solving PDEs we can formulate the problem such that we must solve a system corresponding with a large sparse matrix.

- A sparse matrix is a matrix with most entries equal to 0:

$$\begin{pmatrix} 1.0 & 0 & 5.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.0 & 0 & 0 & 0 & 0 & 11.0 & 0 \\ 0 & 0 & 0 & 0 & 9.0 & 0 & 0 & 0 \\ 0 & 0 & 6.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.0 & 0 & 0 & 0 & 0 \\ 2.0 & 0 & 0 & 0 & 0 & 10.0 & 0 & 0 \\ 0 & 0 & 0 & 8.0 & 0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 & 0 & 0 & 0 & 12.0 \end{pmatrix}$$

- Since there are many zeros it is a waste of time to simply go through every single step in Gaussian elimination to get a solution (normally sparse matrices have thousands of elements or more) so we can use specialised techniques for solving them.

- Many of these involve utilising Krylov subspaces, $\boldsymbol{K_k(A,b)}$, for our sparse matrix equation.

- We essentially have to solve the least squares problem,

$$\min_{\boldsymbol{z}\in K_k(\boldsymbol{A},\boldsymbol{b})} ||\boldsymbol{b}-\boldsymbol{Az}||$$

- The Krylov subspace deals with the zeros so that the algorithm is more efficient.

- Since the minimisation problem depends on the norm, and the norm is defined by the inner product, the algorithm is affected by choice of inner product.

- There is active research which utilises non-standard inner products in order to make this algorithm more efficient.

See: Nonstandard inner products

- There are many applications in probability and statistics for linear algebra however these are not my expertise so I will only give a small glimpse into what can be done.

E.g.

Given a sequence of events whose probabilities depend on the previous event, the probability of a particular sequence appearing in a generation, $t+1$, can be written as the sum of conditional probabilities,

$$p_i(t+1) = P(i|1)p_1(t) + P(i|2)p_2(t) + \cdots + P(i|N)p_N(t)$$

In matrix form we have,

$$\boldsymbol{p}(t+1) = \boldsymbol{A}\boldsymbol{p}(t)$$

To calculate the probability of an event at a given time, T, we have,

$$\boldsymbol{p}(T) = \boldsymbol{A}\boldsymbol{p}(T-1) = \boldsymbol{A}\boldsymbol{A}\boldsymbol{p}(T-2) = \boldsymbol{A}\boldsymbol{A}\boldsymbol{A}\boldsymbol{p}(T-3) = \cdots = \boldsymbol{A}\boldsymbol{A} \ldots \boldsymbol{A}\boldsymbol{p}(T')$$

where T' is some initial time.

So we need to calculate a high power of a matrix which (as shown in practice problems 7) can be done easily via diagonalisation.

3. Methods

Back to contents

This section goes through many (not all) of the methods covered in the lectures with no derivations. Just formulas and quick tips. The lecture notes provide the necessary background as to where the formulas came from and why they work.

- Gaussian/Gauss-Jordan elimination
- Understanding solutions from matrix forms
- Inverse of elementary matrix
- Product of elementary matrices
- Inverse of a matrix by Gauss-Jordan
- Inverse using adjugate
- LU-factorisation
- Solving systems with LU-factorisation
- Testing for subspaces
- Checking for linear independence
- Finding a basis for a vector space
- Finding the dimension of a vector space
- Finding the null space of a matrix
- Finding the row space of a matrix
- Finding the column space of a matrix
- Finding the rank and nullity of a matrix
- Changing basis
- Finding transition matrices
- Testing something is an inner product
- Norm, distance, angle and orthogonality with inner products
- Orthogonal projection onto vector
- Testing for an orthogonal/orthonormal basis
- Coordinates with orthonormal bases
- Creating orthonormal bases (Gram-Schmidt)
- Testing for orthogonal subspaces
- Finding orthogonal complement
- Projecting onto a subspace
- Least squares
- Image/pre-image of a linear transformation
- Showing whether a function is a linear transformation or not
- Using transformed basis vectors
- Kernel of a linear transformation
- Range of a linear transformation
- Testing for isomorphic vector spaces
- Standard matrix for a linear transformation
- Composition of linear transformations
- Inverse transformations

- Matrix of T relative to B and B′ for square matrix transformations
- Matrix of T relative to B and B' for general transformations
- Finding a similar matrix
- Testing if something is an eigenvector
- Finding eigenvalues for a matrix
- Finding eigenvectors for a matrix
- Dimension of eigenspace
- Testing if a matrix is diagonalisable
- Finding a matrix which diagonalises a given matrix
- Finding a basis such that a transformation has diagonal matrix
- Orthogonally diagonalise a matrix
- Calculate Jordan normal form of a matrix
- Easily compute high powers of a matrix

- Gaussian/Gauss-Jordan elimination Back to methods

  - Can do in your own way if it suits you.

  - A fail-safe method is:

Start top left (pivot 1)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Make 0 below

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

Make 0 below

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

Choose next pivot

Make 0 below

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Make all pivots equal to 1

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Make 0 above all pivots

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- Understanding solutions from matrix forms Back to methods

  - Pretend straight line is variable vector:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 4 \\ 7 & 8 & 9 & 6 \end{array}\right)$$

Homogeneous:

- No need to write augmented matrix since it will all be 0's in the right-most column.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- From this we can tell that,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1x_1 + 0x_2 + 1x_3 = 0$$
$$0x_1 + 2x_2 + 3x_3 = 0$$
$$0x_1 + 0x_2 + 1x_3 = 0$$

- The bottom one means x_3 must be 0.

- Substituting back we know all must be 0 (trivial solution).

- If no free variables then $\boldsymbol{x} = \boldsymbol{0}$ **is** unique solution (known as trivial solution).

- If there is a free variable then there are infinite solutions.

- A row of 0's means a free variable and therefore infinite solutions.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$1x_1 + 0x_2 + 1x_3 = 0$$
$$0x_1 + 2x_2 + 3x_3 = 0$$
$$0x_1 + 0x_2 + 0x_3 = 0$$

- x_3 can be anything and we choose x_1 and x_2 as functions of x_3.

Another example:

Pivot

Pivot

$$\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot

This column, x_3, is free (not pivot)

Non-Homogeneous:

- Unique solution has reduced echelon form equal to the identity matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

  The determinant will also be non-zero.

- Inconsistent means no solution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

  The determinant will be zero.

- Infinite solutions:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  The determinant will be zero.

Summary

- Homogeneous can have only unique or infinite solutions.
- Non-homogeneous can have unique, infinite, or no solution.
- Homogeneous always has at least unique solution. This solution is always the zero vector.
- A row of 0's means infinite solutions.
- Be careful for non-homogeneous since both sides of vertical line must be 0 for infinite solutions.
- Choose pivots as basic variables. Non-pivot positions are free variables.

- Inverse of elementary matrix

Back to methods

- Just ask "what matrix do I need to turn it back into the identity matrix?"

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

- This matrix adds 2 times row 1 to row 3.
- How do we get $\boldsymbol{E}$ back to the identity matrix?
- In this case we subtract 2 times row 1 from row 3:

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

➔ $\boldsymbol{E}\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- This matrix multiplies row 2 by -3.
- How do we get $\boldsymbol{E}$ back to the identity matrix?
- In this case we divide row 2 by -3:

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

➔ $\boldsymbol{E}\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- Product of elementary matrices

Back to methods

  - Put in reduced echelon form **one step at a time**.
  - Write down elementary matrices for each step.
  - Find inverses.
  - Multiply together (first matrix on the left).

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}, \qquad \boldsymbol{E_1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{E_3} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E_1^{-1}} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \qquad \boldsymbol{E_2^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \boldsymbol{E_3^{-1}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A} = \boldsymbol{E_1^{-1}} \boldsymbol{E_2^{-1}} \boldsymbol{E_3^{-1}}$$

- Inverse of a matrix by Gauss-Jordan

Back to methods

  - Augment with identity matrix.
  - Make left matrix into identity matrix.
  - The one on the right hand side will be the inverse.

Original matrix → Identity matrix (right)

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

$$\sim\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$$\sim\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

$$\sim\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

Identity matrix (left) — Inverse (right)

- Inverse using adjugate Back to methods

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$|\boldsymbol{A}| = 11$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \\ -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} \\ \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \\ -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} \\ \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} \\ \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} \\ \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \\ \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} & -\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} & \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} & -\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} & \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} & \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{11}\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{11}\begin{pmatrix} 1 & -2 & 5 \\ 6 & -1 & -3 \\ -2 & 4 & 1 \end{pmatrix}$$

- LU-factorisation. Back to methods

  - Reduce to upper triangular matrix, $\boldsymbol{U}$.
  - Write down elementary matrices.
  - Find inverses.
  - $\boldsymbol{L} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1} \dots$ is product of inverses.

$$\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{pmatrix}, \qquad \boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}, \qquad \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

$\boldsymbol{U}$

$$\boldsymbol{E}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{E}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}$$

$$\boldsymbol{L} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}$$

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}$$

- Solving systems with LU-factorisation

Back to methods

$$x_1 - 3x_2 = -5$$
$$x_2 + 3x_3 = -1$$
$$2x_1 - 10x_2 + 2x_3 = -20$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} -5 \\ -1 \\ -20 \end{pmatrix}$$

$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$

$\boldsymbol{y}$

$$\boldsymbol{L} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 2 & -4 & 1 & -20 \end{array}\right) \rightarrow y_1 = -5, y_2 = -1, y_3 = -14$$

$\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$

$\boldsymbol{x}$

$$\boldsymbol{U} = \left(\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 14 & -14 \end{array}\right) \rightarrow x_3 = -1, x_2 = 2, x_1 = 1$$

- Testing for subspaces

Back to methods

  - Quick check → If zero vector is not in the set then NOT a subspace.

$$\{(x, y, 1) : x, y \in \mathbb{R}\}$$

Does NOT contain $(0,0,0)$.

All vectors look like $(0,0,1)$, $(1,2,1), (-1,0,1), etc \ldots$

  - Check 2 things

Closed under addition:

$$S = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

$$(x_1, y_1, 0) + (x_2, y_2, 0)$$

$$= (x_1 + x_2, y_1 + y_1, 0)$$

$$= (w_1, w_2, 0) \in S$$

Of the same form

Closed under scalar multiplication:

$$S = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

$$c(x, y, 0), \qquad c \in \mathbb{R}$$

$$= (cx, cy, 0)$$

$$= (w_1, w_2, 0) \in S$$

Of the same form

Then S is a subspace.

- Checking for linear independence

Back to methods

  - **Solve homogeneous equation.**
  - If only unique solution then linearly independent.

$$S = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So S is linearly independent. ← Unique solution

Remember that the unique solution to the homogeneous equation is ALWAYS the zero vector (trivial solution).

  - If infinite solutions then linearly dependent.

$$S = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So S is linearly dependent. ← Infinite solutions

- Finding a basis for a vector space

Back to methods

  - Find a spanning set.
  - Make linearly independent.
  - Note that a set of n linearly independent vectors spans $\mathbb{R}^n$.

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ → 3 linearly independent vectors → Spans $\mathbb{R}^3$ → It's a basis for $\mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ → 2 linearly independent vectors → Spans $\mathbb{R}^2$

→ Pick 2 linearly independent vectors

→ $S^* = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ spans $\mathbb{R}^2$ and is linearly independent → It's a basis for $\mathbb{R}^2$

- Finding the dimension of a vector space

Back to methods

- Find a basis.
- Count the number of vectors.

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$S^* = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis (still spans the same vector space as S).

There are 2 vectors so $\dim(S) = 2$.

The standard basis for $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$.

There are 3 vectors so $\dim(\mathbb{R}^3) = 3$.

- Finding the null space of a matrix

Back to methods

  - Solve homogeneous equation.
  - Find a basis for the solution.
  - Null space is the span of the basis.

System:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= -1 \\ 7x_1 + 8x_2 + 9x_3 &= 2 \end{aligned}$$

Homogeneous version:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \boxed{\boldsymbol{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

Solution

Basis for solution is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

Null space is $Span \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

- Finding the row space of a matrix

Back to methods

  - Put into reduced echelon form.
  - Non-zero rows are basis for row space.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Non-zero rows

Row space is $Span\{(1,0,-1),(0,1,2)\}$

- Finding the column space of a matrix

Back to methods

  - Put in reduced echelon form.
  - Get linearly independent column numbers.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 3 & 1 & 3 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Linearly independent columns

  - Use those column numbers of the original matrix to make a basis.

Choose same columns of original matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}$$

Column space is $Span\left\{\begin{pmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{pmatrix}\right\}$

- Finding the rank and nullity of a matrix

Back to methods

  - Find a basis for the row or column space → dimension is rank.
  - Find a basis for the null space → dimension is nullity.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Basis for solution is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ ← $\text{nullity} = \dim(nul(\boldsymbol{A})) = \boxed{1}$

2 linearly independent rows/columns ← $\text{rank} = \dim(row(\boldsymbol{A})) = dim(col(\boldsymbol{A})) = \boxed{2}$

Number of columns in matrix is $\boxed{3}$ ←

$\text{rank} + \text{nullity} = \boxed{2} + \boxed{1} = \boxed{3}$

- Changing basis Back to methods

  - Get vector in terms of standard basis.
  - Write as matrix equation
    - Matrix columns formed from basis vectors.
    - Augment with standard basis coordinates.
  - Coordinates in basis, B, are written as $[\boldsymbol{x}]_B$
  - Coordinates in basis, B', are written as $[\boldsymbol{x}]_{B'}$
  - Coordinates in the standard basis, S, are written as $[\boldsymbol{x}]_S$ or just $\boldsymbol{x}$.

Choice of notation for standard basis.

$$B = \{(1,-1),(2,1)\}, \quad B' = \{(1,3),(-2,5)\}, \quad S = \{(1,0),(0,1)\}$$

Starting basis

Desired basis

Standard basis

Write $\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]_B$ relative to B'.

We need to go $B \rightarrow S \rightarrow B'$

Standard basis coordinates

$B \rightarrow S$:

$$\boldsymbol{x} = 2 \times \begin{pmatrix} 1 \\ -1 \end{pmatrix} - 1 \times \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$S \rightarrow B'$:

$$\left(\begin{matrix} 1 & -2 \\ 3 & 5 \end{matrix}\middle|\begin{matrix} 0 \\ -3 \end{matrix}\right) \sim \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} -6/11 \\ -3/11 \end{matrix}\right) \rightarrow [\boldsymbol{x}]_{B'} = \begin{pmatrix} -6/11 \\ -3/11 \end{pmatrix}$$

- Finding transition matrices

Back to methods

$B = \{(1,-1),(2,1)\}$, $B' = \{(1,3),(-2,5)\}$

$B \to S$: $\boldsymbol{P} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

$B' \to S$: $\boldsymbol{Q} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$

Use basis vectors as columns

From $B \to B'$

$(\boldsymbol{Q} \mid \boldsymbol{P})$

Row reduce

$(\boldsymbol{I} \mid \boldsymbol{R})$

$$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 2 \\ 3 & 5 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 3/11 & 12/11 \\ 0 & 1 & -4/11 & -5/11 \end{array}\right)$$

$$\boldsymbol{R} = \begin{pmatrix} 3/11 & 12/11 \\ -4/11 & -5/11 \end{pmatrix}$$

Transition matrix

$[x]_{B'} = \boldsymbol{R}[x]_B$

- Testing something is an inner product Back to methods

  - Can swap order:

$$< \boldsymbol{u}, \boldsymbol{v} > = < \boldsymbol{v}, \boldsymbol{u} >$$

  - Can split up:

$$< \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} > = < \boldsymbol{u}, \boldsymbol{v} > + < \boldsymbol{u}, \boldsymbol{w} >$$

  - Can factor scalar:

$$c < \boldsymbol{u}, \boldsymbol{v} > = < c\boldsymbol{u}, \boldsymbol{v} > = < \boldsymbol{u}, c\boldsymbol{v} >$$

  - Product with self is positive:

$$< \boldsymbol{u}, \boldsymbol{u} > \geq 0$$

  - Only equals 0 if $\boldsymbol{u}$ is zero vector.

$$< \boldsymbol{u}, \boldsymbol{u} > = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$$

Test $< \boldsymbol{u}, \boldsymbol{v} > = u_1 v_1 + 2u_2 v_2$.

1) $< \boldsymbol{u}, \boldsymbol{v} > = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = < \boldsymbol{v}, \boldsymbol{u} >$
2) $< \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} > = u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$
$= u_1 v_1 + 2u_2 v_2 + u_1 w_1 + 2u_2 w_2$
$= < \boldsymbol{u}, \boldsymbol{v} > + < \boldsymbol{u}, \boldsymbol{w} >$
3) $c < \boldsymbol{u}, \boldsymbol{v} > = c(u_1 v_1 + 2u_2 v_2) = (cu_1 v_1 + 2cu_2 v_2)$
$= ((cu_1)v_1 + 2(cu_2)v_2)$
$= < c\boldsymbol{u}, \boldsymbol{v} >$
4) $< \boldsymbol{u}, \boldsymbol{u} > = u_1 u_1 + 2u_2 u_2 = u_1^2 + 2u_2^2 \geq 0$
5) $< \boldsymbol{u}, \boldsymbol{u} > = 0 \Rightarrow u_1^2 + 2u_2^2 = 0 \Rightarrow u_1 = u_2 = 0$

So it is an inner product by definition.

- Norm, distance, angle and orthogonality with inner products

Back to methods

  - Apply definition of inner product as in previous example to the following formulas:

Norm $||\boldsymbol{u}|| = \sqrt{< \boldsymbol{u}, \boldsymbol{u} >}$

Distance $||\boldsymbol{u} - \boldsymbol{v}||$

Angle $\cos\theta = \frac{<\boldsymbol{u},\boldsymbol{v}>}{||\boldsymbol{u}||\,||\boldsymbol{v}||}$

Orthogonal $< \boldsymbol{u}, \boldsymbol{v} \geq 0$

Note that $< \boldsymbol{u}, \boldsymbol{u} >= ||\boldsymbol{u}||^2$

- Orthogonal projection onto vector

Back to methods

  - Using dot product.

Projection of $\boldsymbol{u}$ onto $\mathbf{v}$.

$$proj_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}\boldsymbol{v}$$

Project $\boldsymbol{u} = (4,2)$ onto $\boldsymbol{v} = (3,4)$

$$proj_{\boldsymbol{v}}\boldsymbol{u} = \frac{(4,2)\cdot(3,4)}{(3,4)\cdot(3,4)}(3,4) = \left(\frac{12}{5},\frac{16}{5}\right)$$

  - Using general inner product.

Projection of $\boldsymbol{u}$ onto $\mathbf{v}$.

$$proj_{\boldsymbol{v}}\boldsymbol{u} = \frac{<\boldsymbol{u},\boldsymbol{v}>}{<\boldsymbol{v},\boldsymbol{v}>}\boldsymbol{v} = \frac{<\boldsymbol{u},\boldsymbol{v}>}{||\boldsymbol{v}||^2}\boldsymbol{v}$$

Project $\boldsymbol{u} = (4,2)$ onto $\boldsymbol{v} = (3,4)$ using $<\boldsymbol{u},\boldsymbol{v}> = u_1v_1 + 2u_2v_2$

$$proj_{\boldsymbol{v}}\boldsymbol{u} = \frac{<(4,2),(3,4)>}{<(3,4),(3,4)>}(3,4) = \frac{4\times 3 + 2(2\times 4)}{3\times 3 + 2(4\times 4)}(3,4) = \left(\frac{84}{41},\frac{112}{41}\right)$$

- Testing for an orthogonal/orthonormal basis Back to methods

  - Inner product of each vector combination should be 0.

$$S = \{(1,0,0), (0,1,0), (0,0,01)\}$$

(using dot product in this example but could be general inner product)

$$(1,0,0) \cdot (0,1,0) = 0$$
$$(1,0,0) \cdot (0,0,1) = 0$$
$$(0,1,0) \cdot (0,0,1) = 0$$

So S is orthogonal.

  - Check if unit vectors also.

$$||(1,0,0)|| = \sqrt{1^2 + 0^2 + 0^2} = 1$$
$$||(0,1,0) = \sqrt{0^2 + 1^2 + 0^2} = 1$$
$$||(0,0,1)|| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

So S is orthonormal.

- Coordinates with orthonormal bases

Back to methods

  - Take inner product of vector with basis vectors.

$$B = \left\{\left(\frac{3}{5},\frac{4}{5},0\right),\left(-\frac{4}{5},\frac{3}{5},0\right),(0,0,1)\right\}$$

$v_1 \quad v_2 \quad v_3$

Orthonormal basis

Find $\boldsymbol{w} = (5,-5,2)$ relative to B.

Vector in standard basis

$$c_1 =< \boldsymbol{w}, \boldsymbol{v_1} >$$

$$c_2 =< \boldsymbol{w}, \boldsymbol{v_2} >$$

$$c_3 =< \boldsymbol{w}, \boldsymbol{v_3} >$$

(Using dot product but could be general inner product)

$$c_1 = (5,-5,2)\cdot\left(\frac{3}{5},\frac{4}{5},0\right) = -1$$

$$c_2 = (5,-5,2)\cdot\left(-\frac{4}{5},\frac{3}{5},0\right) = -7$$

$$c_3 = (5,-5,2)\cdot(0,0,1) = 2$$

$$[\boldsymbol{w}]_B = \begin{pmatrix} -1 \\ -7 \\ 2 \end{pmatrix}$$

- Creating orthonormal bases (Gram-Schmidt)

Back to methods

  - Start with a basis and choose first vector:

$$B = \{(1,1,0), (1,2,0), (0,1,2)\}$$

$$\boldsymbol{w_1}$$

  - Make next orthonormal vector using $\boldsymbol{w_1}$ and projecting the second basis vector:

$$\boldsymbol{w_2} = (1,2,0) - proj_{\boldsymbol{w_1}}(1,2,0)$$

  - Make next orthonormal vector using $\boldsymbol{w_1}$, $\boldsymbol{w_2}$ and projecting the third basis vector:

$$\boldsymbol{w_3} = (0,1,2) - proj_{\boldsymbol{w_1}}(0,1,2) - proj_{\boldsymbol{w_2}}(0,1,2)$$

  - Normalise each of the vectors by dividing by their norms.

$$\boldsymbol{u_1} = \frac{\boldsymbol{w_1}}{||\boldsymbol{w_1}||}, \quad , \boldsymbol{u_2} = \frac{\boldsymbol{w_2}}{||\boldsymbol{w_2}||}, \quad , \boldsymbol{u_3} = \frac{\boldsymbol{w_3}}{||\boldsymbol{w_3}||}$$

  - Orthonormal basis will be $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

You can keep following this pattern if there are more basis vectors.

- Testing for orthogonal subspaces

Back to methods

  - Same as testing for orthogonal sets → Inner product of every vector combination between subspaces should be 0.

Subspace 1: $S_1 = Span\{(1,0,1),(1,1,0)\}$

Subspace 2: $S_2 = Span\{(-1,1,1)\}$

Remember all spanning sets are subspaces

Are they orthogonal?

$\boldsymbol{v} \in S_1 \rightarrow \boldsymbol{v} = a(1,0,1) + b(1,1,0)$

$\boldsymbol{w} \in S_2 \rightarrow \boldsymbol{w} = c(-1,1,1)$

Just linear combinations of basis vectors

Where a, b, c are constants.

So $\boldsymbol{v}$ and $\boldsymbol{w}$ represent all vectors in the 2 subspaces.

Try their dot product:

$$\begin{aligned}\boldsymbol{v} \cdot \boldsymbol{w} &= [a(1,0,1) + b(1,1,0)] \cdot [c(-1,1,1)] \\ &= ac(1,0,1) \cdot (-1,1,1) + bc(1,1,0) \cdot (-1,1,1) \\ &= 0\end{aligned}$$

Therefore the subspaces are orthogonal.

- Finding orthogonal complement

Back to methods

  - Create matrix equation so dot product of each vector with $\boldsymbol{x}$ is 0.

Find orthogonal complement of $Span\{(1,2,1,0),(0,0,0,1)\}$.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution to this gives orthogonal complement

Find orthogonal complement of the subspace spanned by the columns of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solution to this gives orthogonal complement

- Projecting onto a subspace Back to methods

  - Find orthonormal basis for the subspace.
  - Projection is the sum of the coordinates multiplied by the corresponding basis vectors.

(orthonormal basis for a subspace) $B = \left\{\left(\frac{3}{5}, \frac{4}{5}, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0\right)\right\}$

Projection of $\boldsymbol{w} = (5, -5, 2)$ onto B:

Coordinates are:

$$c_1 = (5, -5, 2) \cdot \left(\frac{3}{5}, \frac{4}{5}, 0\right) = -1$$

$$c_2 = (5, -5, 2) \cdot \left(-\frac{4}{5}, \frac{3}{5}, 0\right) = -7$$

$$[\boldsymbol{w}]_B = \binom{-1}{-7}$$

Projection onto subspace spanned by B is,

$$-1 \times \left(\frac{3}{5}, \frac{4}{5}, 0\right) - 7 \times \left(-\frac{4}{5}, \frac{3}{5}, 0\right) = (5, -5, 0)$$

- Least squares Back to methods

Find the line of best fit (regression line) of the points (1,0), (2,1), (3,3).

- Assume linear graph can fit:

$$y = ax + b$$

- Substitute points into straight line equation:

$$0 = 1a + b$$
$$1 = 2a + b$$
$$3 = 3a + b$$

- Write in matrix equation:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$\boldsymbol{A}$, $\boldsymbol{X}$, $\boldsymbol{b}$

Be careful here…

b and $\boldsymbol{b}$ are totally different

- Solve $\boldsymbol{A^T A X} = \boldsymbol{A^T b}$

$$\boldsymbol{A^T A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

$$\boldsymbol{A^T b} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}$$

$$\boldsymbol{A^T A X} = \boldsymbol{A^T b} \rightarrow \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} \rightarrow a = \frac{3}{2}, \; b = -\frac{5}{3}$$

$$y = \frac{3x}{2} - \frac{5}{3}$$

- Image/pre-image of a linear transformation Back to methods

$$T = (v_1 - v_2, v_1 + 2v_2) \quad \rightarrow \quad \boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

v_1 v_2

- Find image of $\boldsymbol{v} = (-1,2)$ by applying transformation:

$$T(-1,2) = (-1-2, -1+4) = (-3,3)$$

- Find pre-image of $\boldsymbol{w} = (-1,11)$ by solving linear system:

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 2 & 11 \end{array}\right) \rightarrow \text{Solve} \rightarrow \text{Pre-image is } (3,4)$$

- Showing whether a function is a linear transformation or not Back to methods

$$T(\boldsymbol{u}) = u_1 + u_2$$

- Show can split with addition.

$$T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$$

$$T(\boldsymbol{u}+\boldsymbol{v}) = (u_1 + v_2) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2) = T(\boldsymbol{u}) + T(\boldsymbol{v})$$

- Show can factor scalar.

$$T(c\boldsymbol{u}) = cT(\boldsymbol{u})$$

$$T(c\boldsymbol{u}) = cu_1 + cu_2 = c(u_1 + u_2) = cT(\boldsymbol{u})$$

So T is a linear transformation.

- Using transformed basis vectors

Back to methods

$$T(1,0) = (2,3), \qquad T(0,1) = (-1,1)$$

Basis vectors

Transform $\boldsymbol{v} = (5,4)$.

- Write using basis vectors:

$$(5,4) = 5(1,0) + 4(0,1)$$

$$T(5,4) = 5T(1,0) + 4T(0,1) = 5(2,3) + 4(-1,1) = (6,19)$$

- Kernel of a linear transformation Back to methods

  - Find pre-image of the zero vector.
  - Same as null space for matrix transformations.

$$T(\boldsymbol{v}) = \boldsymbol{Av}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix}$$

Kernel is null space:

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{Solve} \rightarrow \ \boldsymbol{x} = (x_3, -x_3, x_3) = x_3(1, -1, 1)$$

So kernel is $Span\{1, -1, 1\}$

- Range of a linear transformation

Back to methods

  - All images of domain vectors.
  - Same as column space for matrix transformations.

$$T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix}$$

Find column space:

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

First 2 columns are linearly independent so we use:

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix}$$

$$col(\boldsymbol{A}) = Span\{(1,-1),(-1,2)\}$$

This is the range

- Testing for isomorphic vector spaces Back to methods

    - Show dimension of domain = dimension of codomain

If we transform from one vector space to the other we might have:

Vector space 1: $V = \mathbb{R}^3$

Vector space 2: $W = P_2$ (polynomials of degree 2 or less)

Are they ismorphic?

- To get dimensions we find a basis. Let's use the standard bases for these vector spaces:

$$B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$B_2 = \{1, x, x^2\}$$

The number of basis vectors in both vector spaces is 3.

Therefore their dimensions are both 3.

<u>Therefore they are isomorphic</u> (they essentially store information in the same way).

- Standard matrix for a linear transformation Back to methods

  - We just put the coefficients of each variable as the columns of the matrix.

$$T(x, y, z) = (2x - y, 3y - 2z, x + 2y + 3z)$$

Has standard matrix:

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 2 & 3 \end{pmatrix}$$

- Composition of linear transformations [Back to methods]

  - Apply transformations one after the other.
  - Same as multiplying matrices for matrix transformations.

$\mathbb{R}^3 \to \mathbb{R}^2$

$$T_1(x, y, z) = (x - y, y + z)$$

$\mathbb{R}^2 \to \mathbb{R}^4$

$$T_2(x, y) = (2x - y, x + y, -x + 2y, -2x + 3y)$$

The codomain dimension of the first transformation must match the domain of the second transformation.

In this case we have:

$$T(\boldsymbol{v}) = [T_2 \circ T_1](\boldsymbol{v})$$

Apply T_1 first then T_2

➔ $\boldsymbol{A_1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

➔ $\boldsymbol{A_2} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \\ -2 & 3 \end{pmatrix}$

$\mathbb{R}^3 \to \mathbb{R}^4$

$$\boldsymbol{A} = \boldsymbol{A_2 A_1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -1 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \\ -2 & 5 & 3 \end{pmatrix}$$

Note that,

$$T(\boldsymbol{v}) = [T_1 \circ T_2](\boldsymbol{v})$$

Is undefined because the dimensions don't match.

- Inverse transformations Back to methods

  - For matrix transformations just find inverse matrix.

$$T(x, y) = (x + 2y, 3x + 4y)$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

➔ $\boldsymbol{A}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

  - Write as transformation:

$$T^{-1}(\boldsymbol{v}) = \left(-2x + y, \frac{3x}{2} - \frac{y}{2}\right)$$

- Matrix of T relative to B and B' for square matrix transformations Back to methods

  - Trying to find $\boldsymbol{A}^*$:

$$\boldsymbol{v} \longrightarrow T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v} = \boldsymbol{w}$$

Find transformation between 2 different bases that achieves the same thing but with different coordinates:

$$[\boldsymbol{v}]_{\boldsymbol{B}} \longrightarrow \boldsymbol{T}[\boldsymbol{v}]_{\boldsymbol{B}} = \boldsymbol{A}^*[\boldsymbol{v}]_{\boldsymbol{B}} = [\boldsymbol{w}]_{\boldsymbol{B'}}$$

Start in basis, B, finish in basis B'

So it is the same transformation but the input and output have just been written in 2 different bases.

  - Use formula:

$$\boldsymbol{A}^* = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{P}$$

Where the transition matrices are,

$\boldsymbol{P} : B \rightarrow S$ ← $\boldsymbol{P}$ is transition matrix from B to the standard basis

$\boldsymbol{Q} : B' \rightarrow S$ ← $\boldsymbol{Q}$ is transition matrix from B' to the standard basis

---

E.g.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$ ← Relative to standard basis

How about between these 2 bases?

$$B = \{(1,2), (-1,1)\}, \qquad B' = \{(-1,3), (2,1)\}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \qquad \boldsymbol{Q} = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\boldsymbol{Q}^{-1} = \begin{pmatrix} -1/7 & 2/7 \\ 3/7 & 1/7 \end{pmatrix} \rightarrow \boldsymbol{A}^* = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -3/7 & -6/7 \\ 9/7 & -3/7 \end{pmatrix}$$

Matrix of T relative to B and B'

- Matrix of T relative to B and B' for general transformations

Back to methods

  - Use this method when the matrix is either square or not square.
  - Transform basis vectors in B, then write with respect to B'.

Let's do this technique on the last example to check we get the same thing:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$B = \{(1,2), (-1,1)\}, \qquad B' = \{(-1,3), (2,1)\}$$

Transform basis vectors of B:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Write in terms of B':

$$\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{9}{7} \end{array}\right) \rightarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix} = -\frac{3}{7}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + \frac{9}{7}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right]_{B'} = \begin{pmatrix} -3/7 \\ 9/7 \end{pmatrix}$$

Similarly we have $\left[\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right]_{B'} = \begin{pmatrix} -6/7 \\ -3/7 \end{pmatrix}$

$$\boldsymbol{A}^* = \begin{pmatrix} -3/7 & -6/7 \\ 9/7 & -3/7 \end{pmatrix}$$

Remember we can use this method even when the matrix is not square.

- Finding a similar matrix Back to methods

    - Find the transition matrix between the bases, $\boldsymbol{R}$.
    - Calculate its inverse, $\boldsymbol{R^{-1}}$.
    - Apply the formula:

$$\boldsymbol{A'} = \boldsymbol{R^{-1}AR}$$

The following matrix is a transformation with respect to the basis, B.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

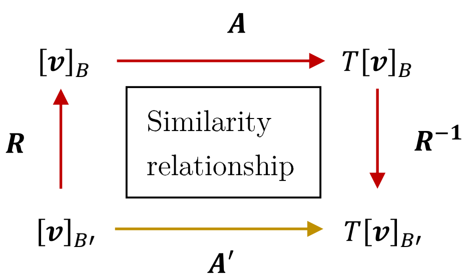
$$B = \{(1,2), (-1,1)\},$$

Calculate a similar matrix that represents the same transformation but in the basis, B'.

$$B' = \{(-1,3), (2,1)\}$$

$\boldsymbol{P} : B \to S$ ← $\boldsymbol{P} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$\boldsymbol{Q} : B' \to S$ ← $\boldsymbol{Q} = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$



We are looking for $\boldsymbol{A'}$.

First we get $\boldsymbol{R}$: $\overrightarrow{B' \to B}$ → $\overleftarrow{(\boldsymbol{P} \mid \boldsymbol{Q})}$

$$\underset{\boldsymbol{P}}{\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}} \left| \underset{\boldsymbol{Q}}{\begin{matrix} -1 & 2 \\ 3 & 1 \end{matrix}} \right) \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left| \underset{\boldsymbol{R}}{\begin{matrix} 2/3 & 1 \\ 5/3 & -1 \end{matrix}} \right)$$

$$\boldsymbol{R} = \begin{pmatrix} 2/3 & 1 \\ 5/3 & -1 \end{pmatrix}, \qquad \boldsymbol{R^{-1}} = \begin{pmatrix} 3/7 & 3/7 \\ 5/7 & -2/7 \end{pmatrix}$$

$$\boldsymbol{A'} = \boldsymbol{R^{-1}AR} = \begin{pmatrix} 38/7 & -6/7 \\ 8/21 & -3/7 \end{pmatrix}$$

- Testing if something is an eigenvector Back to methods

  - Substitute into the eigenvalue problem and check if the answer is a multiple of the input.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Is $\boldsymbol{x} = (1,0)$ is an eigenvector?

This is the eigenvalue

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Answer is a multiple of input vector therefore it is an eigenvector

- Finding eigenvalues for a matrix

Back to methods

  - Solve,

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

Find the eigenvalues of,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

➔ $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 2$

➔ $\lambda_1 = \frac{5}{2} + \frac{\sqrt{33}}{2}, \qquad \lambda_2 = \frac{5}{2} - \frac{\sqrt{33}}{2}$

2 distinct eigenvalues

- Finding eigenvectors for a matrix

Back to methods

  - First get eigenvalues.
  - Substitute eigenvalues into $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ and solve.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$$

Matrix is <u>triangular</u> so eigenvalues are on <u>leading diagonal</u>

$\lambda_1 = 1$:

$$\begin{pmatrix} 1-1 & 2 \\ 0 & 4-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Choose any x_1 we like, for example $x_1 = 1$ → $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector.

$\lambda_1 = 4$:

$$\begin{pmatrix} 1-4 & 2 \\ 0 & 4-4 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{x} = x_2 \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

Choose any x_2 we like, for example $x_2 = 3$ → $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector.

- Dimension of eigenspace

Back to methods

  - We have an eigenspace for each eigenvalue.
  - Get the eigenvectors in the usual way then count how many linearly independent ones we have.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{pmatrix}$$

Has eigenvalues, $\lambda = -3, 3$.

$\lambda_1 = -3$:

$$\begin{pmatrix} 1+3 & 2 & -3 \\ -2 & 5+3 & -2 \\ -6 & 6 & -3+3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -3 \\ -2 & 8 & -2 \\ -6 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{x} = x_3 \begin{pmatrix} 1/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \text{ is an eigenvector } \rightarrow \dim(E_{\lambda_1}) = 1$$

$\lambda_1 = 3$:

$$\begin{pmatrix} 1-3 & 2 & -3 \\ -2 & 5-3 & -2 \\ -6 & 6 & -3-3 \end{pmatrix} = \begin{pmatrix} -2 & 2 & -3 \\ -2 & 2 & -2 \\ -6 & 6 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{x} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ are linearly independent eigenvectors } \rightarrow \dim(E_{\lambda_2}) = 2$$

- Testing if a matrix is diagonalisable

Back to methods

  - No. of linearly independent eigenvectors = size of square matrix.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{pmatrix}$$

Has 3 linearly independent eigenvectors:

$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

And it is a 3×3 matrix → $\boldsymbol{A}$ is diagonalisable.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Has only 1 eigenvector:

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

And it is a 2×2 matrix → $\boldsymbol{A}$ is not diagonalisable.

- Finding a matrix which diagonalises a given matrix

Back to methods

  - Make a matrix with columns which are the eigenvectors.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{pmatrix}$$

Eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \boldsymbol{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \ \boldsymbol{P^{-1}} = \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 4/3 & -1/3 \\ -1 & 1 & 0 \end{pmatrix}$

Diagonal matrix is:

$$\boldsymbol{D} = \boldsymbol{P^{-1}AP} = \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 4/3 & -1/3 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & -3 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The diagonal entries are just the eigenvalues

- Finding a basis such that a transformation has diagonal matrix Back to methods

  - The basis is just the linearly independent eigenvectors that diagonalise the transformation matrix.

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}$$

Eigenvalues are $\lambda = 2, -2, 3$.

Same procedure as previous example to get eigenvectors:

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

A basis such that the transformation matrix is diagonal is just the set of these vectors.

$$B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

- Orthogonally diagonalise a matrix

Back to methods

- Check if matrix is symmetric.
- Normalise any eigenvectors with corresponding eigenvalue of multiplicity 1.
- Use Gram-Schmidt for multiplicity > 1.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$$

Eigenvalues are $\lambda = -2, 3$.

Eigenvectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

Both multiplicity 1 so already are orthogonal: $(1,-2) \cdot (-2,-1) = 0$

Just normalise to get

$$\left\{\frac{1}{\sqrt{5}}(1,-2), \frac{1}{\sqrt{5}}(-2,-1)\right\}$$

Orthonormal set

➔ $\boldsymbol{P} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}$, $\boldsymbol{P}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}$

➔ $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$$

Eigenvalues are $\lambda_1 = -6$, $\lambda_2 = 3$

Multiplicity 2
→ Gram-Schmidt

Eigenvectors: $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

Gram-Schmidt: $\boldsymbol{w_2} = (2,1,0)$

$$\boldsymbol{w_3} = (-2,0,1) - proj_{\boldsymbol{w_2}}(-2,0,1) = (-2,0,1) - \frac{(-2,0,1)\cdot(2,1,0)}{||(2,1,0)||^2}(2,1,0) = \left(-\frac{2}{5}, \frac{4}{5}, 1\right)$$

$\boldsymbol{u_2} = \frac{\boldsymbol{w_2}}{||\boldsymbol{w_2}||}$, $\boldsymbol{u_3} = \frac{\boldsymbol{w_3}}{||\boldsymbol{w_3}||}$

$\boldsymbol{u_1} = \frac{1}{3}(1,-2,2)$

Orthonormal set is $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$

Make matrix columns from these vectors

- Calculate Jordan normal form of a matrix

Back to methods

- Find eigenvalues.
- Get linearly independent eigenvectors.
- Use generalised eigenvectors if necessary to make $\boldsymbol{T} \rightarrow \boldsymbol{J} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$.

$$\boldsymbol{A} = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 7 & -16 & -48 & 46 \\ -6 & 16 & 43 & -38 \\ -3 & 9 & 23 & -19 \end{pmatrix}$$

Eigenvalues are $\lambda = 3$ ← Multiplicity 4

Rank 1 eigenvectors: $(\boldsymbol{A} - 3\boldsymbol{I})$ → eigenvectors are of the form $\begin{pmatrix} -1 \\ -3 \\ 2 \\ 1 \end{pmatrix}$

Rank 2 eigenvectors: $(\boldsymbol{A} - 3\boldsymbol{I})^2$ → eigenvectors are of the form $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ 2 \\ 0 \end{pmatrix}$

Rank 3 eigenvectors: $(\boldsymbol{A} - 3\boldsymbol{I})^3$ → eigenvectors are of the form $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Rank 4 eigenvectors: $(\boldsymbol{A} - 3\boldsymbol{I})^4$ → eigenvectors are of the form $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Choose this as initial eigenvector

(can choose any as long as it is linearly independent from the other rank eigenvectors)

$$\boldsymbol{x}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{x}_3 = (\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{x}_4 = \begin{pmatrix} 1 \\ 7 \\ -6 \\ -3 \end{pmatrix}$$

$$\boldsymbol{x}_2 = (\boldsymbol{A} - 3\boldsymbol{I})^2\boldsymbol{x}_4 = \begin{pmatrix} 6 \\ 24 \\ -20 \\ -12 \end{pmatrix}$$

$$\boldsymbol{x}_1 = (\boldsymbol{A} - 3\boldsymbol{I})^3\boldsymbol{x}_4 = \begin{pmatrix} -2 \\ -6 \\ 4 \\ 2 \end{pmatrix}$$

$$\boldsymbol{T} = \begin{pmatrix} -2 & 6 & 1 & 1 \\ -6 & 24 & 7 & 0 \\ 4 & -20 & -6 & 0 \\ 2 & -12 & -3 & 0 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 0 & -3/4 & -3/4 & -1/4 \\ 0 & 0 & 1/4 & -1/2 \\ 0 & -1/2 & -3/2 & 3/2 \\ 1 & -1 & -3/2 & 1 \end{pmatrix}$$

$$J = T^{-1}AT = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

- Easily compute high powers of a matrix Back to methods

  - Diagonalise the matrix.
  - Raise the diagonals to the desired power then use similarity.

$$\boldsymbol{A} = \begin{pmatrix} 10 & 18 \\ -6 & -11 \end{pmatrix}, \text{ find } \boldsymbol{A}^6$$

Eigenvectors are $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

➔ $\boldsymbol{P} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$, $\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$

Diagonalise: $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 10 & 18 \\ -6 & -11 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

Raise to power: $\boldsymbol{B}^6 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}^6 = \begin{pmatrix} (-2)^6 & 0 \\ 0 & 1^6 \end{pmatrix} = \begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix}$

Similarity: $\boldsymbol{A}^6 = \boldsymbol{P}\boldsymbol{B}^6\boldsymbol{P}^{-1} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -188 & -378 \\ 126 & 253 \end{pmatrix}$

4. Glossary

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This section contains key terms and definitions in plain English. Note that plain English descriptions are good for understanding but can sometimes lead to mistakes if used with no regard for the rigourous mathematical definitions.

Adjugate	Transpose of the cofactor matrix. Used for calculating inverse matrices.
Angle	A measure of the difference in direction of vectors. Given by the cosine formula.
Augmented matrix	Extending the linear system matrix with either a column vector or another matrix.
Basic variable	Variable corresponding with the pivot positions in the echelon form. We write these variables in terms of the free variables. They are like dependent variables.
Basis	A linearly independent spanning set.
Block matrix	A matrix made up of other smaller matrices.
Cauchy-Schwarz Inequality	Says that the absolute value of the inner product of 2 vectors is less than or equal to the norms of the 2 vectors multiplied together.
Characteristic equation	The equation defined by a determinant that gives us the eigenvalues of a matrix.
Characteristic polynomial	The polynomial that comes from expanding the characteristic equation.
Closed under addition	Taking 2 vectors from a set and adding them produces another vector in that set.
Closed under scalar multiplication	Taking a vector in a set and multiplying it by a scalar produces another vector in the set.
Codomain	The set of possible outcome values of a function.
Coefficient	The number multiplying a variable.
Column space	The set of all possible solutions to a matrix equation. It is the span of the column vectors of the matrix.

Column vector	A vertical vector. $(1 \times n)$ matrix.
Composite transformation	Combining 2 separate transformations into a single step.
Coordinates	The number multiplying the basis vectors. This describes the location of a point.
Diagonal matrix	Zeroes everywhere except the leading diagonal.
Diagonalisation	The process of finding a similar matrix that is diagonal to a given matrix.
Dimension	Number of basis vectors of a vector space.
Distance	This is the norm of the difference between 2 vectors. Can be defined with a general inner product or standard dot product.
Domain	Set of input values.
Dot product	Multiplication of 2 vectors that produces a vector output as defined in elementary mathematics courses.
Echelon form	Special form of a matrix that can be used to solve a linear system using back-substitution.
Eigenspace	The set of all possible eigenvectors for a given eigenvalue.
Eigenvalue	A number corresponding with the eigenvalue problem. It is associated with an eigenspace. The effect is to stretch or compress the corresponding eigenvectors.
Eigenvector	A vector corresponding with the eigenvalue problem. Its direction is unchanged by the matrix of transformation. It is stretched by a factor equal to the eigenvalue.
Elementary matrix	A matrix which is one row operation away from the identity matrix.
Euclidean inner product	The standard for product.
Free variable	A variable corresponding with a non-pivot position of an echelon form. It is like an independent variable (we can choose it).
Gaussian elimination	The process of obtaining an echelon form of a matrix.
Gauss-Jordan elimination	The process of obtaining the reduced echelon form of a matrix.

Generalised eigenvector	A vector which solves the eigenvalue equation raised to a power.
Gram-Schmidt process	The process of obtaining an orthonormal basis from a general basis.
Homogeneous system	A system with no constant terms. $$\boldsymbol{Ax} = \boldsymbol{0}$$ Every term has an $\boldsymbol{x}$ in it.
Identity matrix	Zeroes everywhere except the leading diagonal which has entries equal to 1.
Image	The output of transforming a vector.
Inconsistent	A system which has no solution.
Infinite solutions	An infinite number of solutions to a system. There will always be a free variable.
Inner product	An operation on 2 vectors which obeys 5 rules (sometimes 4^{th} and 5^{th} rule are combined together).
Inner product space	A vector space which we also define an inner product on.
Inverse matrix	The matrix which when multiplied by the original matrix produces the identity matrix.
Inverse transformation	Going back from output to input. For matrix transformations it is just the inverse matrix.
Invertible matrix	A matrix which has an inverse. The determinant is not zero. A system with an invertible matrix has a unique solution.
Isomorphism	Vector spaces which essentially store the same information but in a different format. Technically it is when their dimensions match.
Jordan normal form	A unique form of any matrix. It is a block matrix corresponding with the eigenvalues.
Kernel	The set of all vectors that get transformed into the zero vector.
Leading diagonal	The longest diagonal of a square matrix. From top left to bottom right.
Leading entry	The first non-zero entry of a row.
Least squares	A method for finding lines of best fit.

	It approximates a solution to an inconsistent system.
Left-multiplication	Multiplying a matrix on the left hand side.
Linear combination	Adding vectors together which are multiplied by various constants.
Linear equation	Where the variable is not raised to any power or used in transcendental functions (sine, cosine, exp etc.)
Linear operator	A linear transformation (often with function inputs).
Linear system	A collection of linear equations that share variables.
Linear transformation	A function that obeys 2 specific rules to do with addition and scalar multiplication.
Linearly dependent	Vectors which can be written as linear combinations of each other.
Linearly independent	Vectors which cannot be written as linear combinations of each other.
Lower triangular matrix	All entries above the leading diagonal are 0.
LU-factorisation	A type of matrix factorisation where the left factor is a lower triangular matrix and the right factor is an upper triangular matrix.
Matrix equation	$\boldsymbol{Ax} = \boldsymbol{b}$ Represents a linear system of equations or a linear transformation.
Matrix of T relative to B and B'	The matrix of a linear transformation that starts in the basis, B, and gives the output in the basis, B'.
Multiplicity	The number of times an eigenvalue appears as a root to the characteristic polynomial.
Non-homogeneous system	There is a constant term in the matrix equation. $\boldsymbol{Ax} = \boldsymbol{b}, \qquad \boldsymbol{b} \neq \boldsymbol{0}$
Nonlinear equation	The variable is raised to a power or used in a transcendental function. Or if multiplied by another variable.
Nonlinear system	A collection of nonlinear equations that share variables.
Non-standard basis	A basis that is not the usual orthonormal basis for the given vector space.

Norm	Gives an idea of size of a vector. Defined with either the dot product or a general inner product.
Normalise	Create a vector with norm equal to 1 from a general length vector. We do this by dividing the given vector by its own norm.
Null space	The set of all solutions to the homogeneous equation.
Nullity	Dimension of the null space.
One-to-one	Every input of a linear transformation has exactly 1 output.
Onto	Every possible output is achieved (codomain = range).
Operator	Performs some task based on some inputs. Inputs can be numbers, variables, functions etc.
Orthogonal complement	The subspace which has all the vectors that are orthogonal to a given subspace.
Orthogonal diagonalisation	Diagonalising a matrix using an orthogonal matrix.
Orthogonal matrix	A matrix with orthonormal columns and rows.
Orthogonal projection onto subspace	The coordinates of a vector relative to the basis of the subspace.
Orthogonal projection onto vector	The shadow of a vector onto another vector.
Orthogonal sets	Every combination of vectors is orthogonal.
Orthogonal subspaces	Subspaces which form an orthogonal complement.
Orthogonal vectors	Inner product is 0.
Orthonormal	Both orthogonal and of unit length.
Orthonormal basis	A basis whose vectors are an orthonormal set.
Orthonormal set	All vectors are orthogonal to each other and also of unit length.
Particular solution	The solution to the non-homogeneous part of the matrix equation.
Pivot	The leading entry of a row.
Pivot column	The column which contains a pivot.

Pre-image	The input which resulted in the specified output.
Range	The set of outputs for the specific inputs.
Rank	How many linearly independent rows/columns a matrix has.
Reduced echelon form (REF)	Special form of a matrix that can be used to read off solutions to linear systems. Every matrix has a unique REF, but many matrices might have the same REF.
Regression line	Straight line of best fit.
Row operation	Multiplying a row by a constant. Swapping rows. Adding multiples of one row to another. They can also be represented by elementary matrices.
Row space	The set of all linear combinations of the rows of a matrix.
Row vector	A horizontal vector.
Similar matrices	Matrices which represent the same linear transformation but in different bases.
Solution space	The set of all possible solutions to either the homogeneous or non-homogeneous matrix equation.
Span/Spanning set	The set of all possible linear combinations of a set of vectors.
Standard basis	An orthonormal basis for a vector space which has 1's in the corresponding positions for each vector in the basis.
Standard matrix	Matrix of transformation in the standard basis.
Subspace	A subset of a vector space that is closed under addition and scalar multiplication.
Symmetric matrix	A matrix which is equal to its transpose.
Transition matrix	A matrix which you can multiply a vector by to get the same vector in an alternative basis.
Transpose	Changing the columns of a matrix to be the rows and the rows to be the columns.
Triangle inequality	A generalisation of Pythgoras' theorem.
Triangular matrix	Either an upper or lower triangular matrix.

Union	A set containing the elements of the 2 sets being unioned.
Unique solution	Only one set of values satisfies the matrix equation. Corresponds with a zero determinant.
Unit vector	A vector with norm equal to 1.
Upper triangular matrix	A matrix with zeroes below the leading diagonal.
Vector	A mathematical object that has both size and direction.
Vector space	A set of objects that obeys 10 specific rules.
Zero matrix	Every entry of the matrix is 0.
Zero vector	Every entry of the vector is 0.